Convergence Test summary

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The following chart summarizing the primary convergence tests: when they show convergence, when they show divergence, and what assumptions are needed in order to apply them.

Throughout, assume that the series in question is $\sum_{n=a}^{\infty} c_n$.

Test	Converges if...	Diverges if...	Assumptions
Integral	$\int_a^\infty f(x)dx$ converges	$\int_a^\infty f(x)dx$ diverges	$f(x) \geq 0$ and $f(x)$ is decreasing $c_n = f(n)$ for each n
Ratio	$L < 1$	$L > 1$	$L = \lim_{n\to\infty} \left\lvert \frac{c_{n+1}}{c_n} \right\rvert$ exists
Comparison	Smaller than a convergent series	Larger than a divergent series	Positive terms
Alternating series	$\lvert c_n \rvert$ is decreasing and $\lim_{n\to\infty} c_n = 0$	N.A.	Series alternates signs
nth term	N.A.	$\lim_{n\to\infty} c_n \neq 0$	none

We did not formally discuss the "nth term test" in class, because in practice it is the least useful. However, it is worth mentioning since it often gives a very fast way to see that a series is divergent.

1 Choosing which test to apply

The best way to figure out which test to apply is to examine many examples. With enough practice, you will not think in terms of individual "tests" at all, and develop an intuitive sense for what sorts of series converge (and why). That said, when you are first learning, a couple general tips may be useful.

- If the series is an alternating series, check first whether the alternating series test applies. This is often the easiest test to apply.
- If you recognize the terms of the series as coming from a relatively simple function that you think you could integrate, apply the integral test.
- If the terms are products of several relatively simple factors, the computation in the ratio test is likely to be tractable.
- The comparison test is usually only used when the other tests look too cumbersome to apply, but you think you can get to an easier series by "fudging" the terms to something simpler.

2 Remarks

2.1 p-series

A key example, and main application of the integral test, are the so-called "p-series," which are series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (for some constant p). Such a series converges if $p > 1$, and diverges otherwise; this follows from evaluting the integral $\int_1^{\infty} \frac{1}{x^p} dx$. You are free to use (e.g. in the comparison test) this fact without further comment.

One interesting feature of p-series is that the ratio test is useless in studying them, since the limit of the ratio of adjacent terms is exactly 1.

2.2 The case $L = 1$ in the ratio test

If you compute the limit of ratios L, and it is equal to 1, then the ratio test has nothing useful to tell you. Some such series converge, and some diverge. A different test is needed.

2.3 The case $L = \infty$ in the ratio test

When applying the ratio test, you usually need the limit L to converge. However, there is one key exception: if it diverges to infinity, then the ratio test can still be applied, and it guarantees that the series diverges.

2.4 Polynomial factors in the ratio test

One great thing about the ratio test is that any polynomial factors in the numerator or denominator become irrelevant. This is because if $p(x)$ is any polynomial, then

$$\lim_{n \to \infty} \frac{p(n+1)}{p(n)} = 1.$$

This fact is a consequence, for example, of l'Hôpital's rule, but there are other ways to see it as well.

For example, if the series is $\sum_{n=1}^{\infty} \frac{1}{3^n}$, or $\sum_{n=1}^{\infty} n^2 \frac{1}{3^n}$, or $\sum_{n=1}^{\infty} \frac{1}{n^3 + 7n + 1} \frac{1}{3^n}$, then the limit L is $\frac{1}{3}$. The polynomial factors drop out entirely when L is computed. This can vastly simplify the work of computing L, and hence determining whether the series converges.

2.5 The "N.A." entries in the table

The alternating series test does not include a criterion for divergence. It can only show convergence. Similarly, the nth term test cannot be used to show that a series converges. It can only guarantee that it diverges. These two tests can only work in one direction.