Differential Equations

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These notes summarize some basic topics in differential equations, which we will cover this semester in Math 19. These notes necessarily only touch on a couple topics as an introduction to the topic. There are many different kinds of differential equations, and many different techniques. We will emphasize only two types of differential equations that happen to be very well understood: linear homogeneous equations and separable equations.

In algebra, you learn how to solve many types of *algebraic* equations (linear equations, quadratic equations, systems of equations, etc.). The solutions to algebraic equations are *numbers*. Solving equations is the process of going from something you know *about* an unknown number to determining the number itself.

The primary difference between calculus and algebra is that it is not *individual numbers* that you are interested in, but *functions*. For example, instead of wanting to know how many deer there are in a forest today, you would like to know a function (which you could plot in a graph) that estimates how many deer there will be in the forest on *any* future day (in practice, of course, the model is only useful for relatively short windows into the future).

In a differential equation, there is not an unknown *quantity*, but rather an unknown *function*. To solve a differential equation means to go from something you know *about* a function (e.g. that it is equal to its own second derivative) to knowing the function itself.

1 Basic notions and terminology

1.1 Differential equations

A differential equation describes how a quantity evolves over time. The quantity might be a physical measurement (such as the position of a mass moving on a spring or the height of a flying projectile), or it might be something on a larger scale, such as the number of deer in a given population. A differential equation gives a quantitative version of statements like "a force on an object causes the object to accelerate" (where the word "accelerate" indicates that this is a statement about some second derivative), or "the birth rate and death rate in the deer population depends on these several environmental factors" (where the '"birth rate death rate" together tell the rate of change, or first derivative, of the population).

The evolution over time is described using a derivative. This might be a first derivative, a second derivative, or some higher derivative (although most of the equations of classical physics are second or first order). The highest derivative appearing in the equation is called the *order* of the differential equations.

Example 1.1. The following are all *first order* differential equations.

- $f'(t) = f(t)$
- $\dfrac{dy}{dx} = x + y$
- $f'(x) + 5f(x) = 7 + \cos x$

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Example 1.2. The following are all *second order* differential equations.

- $f''(t) = -9f(t)$
- $f''(x) + 7f'(x) + 10f(x) = 0$
- $\dfrac{d^2y}{dx^2} = 5\dfrac{dy}{dx} + y + x$

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1.2 Specific solutions and general solutions

A solution to a differential equation is some particular function that makes the equation hold true.

Example 1.3. Here are some specific solutions to two of the differential equations mentioned above.

- The function $f(t) = e^t$ is a solution to the differential equation $f'(t) = f(t)$, because it is equal to its own derivative.
- The function $y = -x - 1$ is a solution to the differential equation $\frac{dy}{dx} = x + y$.
- The function $f(t) = \cos(3t)$ is a solution to the differential equation $f''(t) = -9f(t)$.

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Usually, a differential equation has many possible solutions. It is often convenient to express all of these possible solutions in a single formula. To do this, it is necessary to include some constants in the formula. Choosing different values of these constants gives different functions, all of which should be solutions to the differential equation.

If you can find an expression, with some constants in it, that expresses *every possible solution* to a differential equation, then this expression is called the *general solution* to the differential equation.

You are already familiar with the use of constants to write down general solutions to differential equations, although you did not call them differential equations at the time.

Example 1.4. Consider the equation

$$f'(t) = \cos t.$$

This is a differential equation of an extremely simple kind: you are given the derivative of the function precisely. All you need to do to solve this equation is to find the antiderivative of $\cos t$, also called the indefinite integral.

$$\begin{aligned} f(t) &= \int \cos t dt \\ f(t) &= \sin t + C \end{aligned}$$

That little symbol you reflexively add at the end of the express, $+C$, is just acknowledging that there isn't just *one* function whose derivative is $\cos t$. There's a whole family of them, which includes $\sin t, \sin t + 7, \sin t - 1$, and so forth. The single expression $f(t) = \sin t + C$ simultaneously expresses all of these possible solutions, so it is called the "general solution." ◁

Example 1.5. Consider the equation

$$f''(t) = -10.$$

Again, you can find the general solution to this equation using methods you already know: simply take the antiderivative twice.

$$\begin{aligned} f'(t) &= \int (-10) dt \\ &= -10t + C \\ f(t) &= \int (-10t + C) dt \\ &= -5t^2 + Ct + D \end{aligned}$$

Here D is a *second* constant. I used the letter D since C was already taken. So the expression $f(t) = -5t^2+Ct+D$ is the "general solution" of the equation $f''(t) = -10$, since it expresses every possible function that satisfies the differential equation. It is not possible to do this with only one constant, because the family of solutions is too large. ◁

Notice that example 1.4 is a first order differential equation, and one constant is necessary to express the general solution, while example 1.5 is a second order differential equation, and two constants are necessary to express the general solution. This is part of a general pattern.

Fact. The general solution of a first order differential equation has one constant in it. The general solution of a second order differential equation has two constants in it. In general, the number of constants in the general solution is equal to the order of the differential equation.

In fact, the reverse is also true.

Fact. If an expression for solutions $f(t)$ to a given order n differential equation has n constants in it, then this expression gives the general solution.

Caveat. Technically speaking, you should make sure that none of the constants in the expression are redundant. For example, the expression $f(t) = C \sin t - D \sin t$ is *not* the general solution to the second order equation $f''(t) = -f(t)$, because the constant D doesn't add anything new. The expression $f(t) = C \sin t$ would express the same class of possible functions.

You may take this fact as given in this class. In a more theoretical course, this fact would need to be stated more precisely and a few caveats would have to be added, but for our purposes this statement will be good enough. If you are interested, the general fact following from the *existence and uniqueness theorem for ordinary differential equations.*

1.3 Initial value problems

The reason that differential equations have many possible solutions is simple: a differential equation tells how something evolves over time, but it doesn't tell you where it started in the first place.

Example 1.6. The differential equation $M'(t) = -M(t)$ could describe the mass of some radioactive element in a fossil (it says the the substance disappears at a rate proportional to the amount of the element present). This tells how the mass evolves over time, but it doesn't tell how much mass was there at the beginning, so this is not enough information to tell what the function $M(t)$ is. ◁

Therefore, to fully specify a function, it isn't enough just to tell how the quantity will evolve over time (the differential equation). It is also necessary to tell where it started (the initial data). Once this initial data is given, it is possible to recover the function uniquely.

An *initial value problem* consists of the following two things.

- A differential equation. (tells how things evolve over time)
- One or more initial conditions. (tells how things began)

Typically, the initial conditions consist of the following data.

- For a first order differential equation, the initial value $f(0)$ is given.
- For a second order differential equation, the initial values $f(0)$ and $f'(0)$ are given.
- In general, for an nth order differential equation, the values $f(0), f'(0), \cdots, f^{(n-1)}(0)$ are given.

Example 1.7. The following is a first order initial value problem.

$$\begin{aligned} f'(x) &= 5f(x) \\ f(0) &= 7 \end{aligned}$$

This initial value problem has a unique solution. In fact, this solution is $f(x) = 7e^{5x}$ (we'll see how to find this solution later). ◁

Example 1.8. The following is a second order initial value problem.

$$\begin{aligned} f''(x) &= -f(x) \\ f(0) &= 5 \\ f'(0) &= -7 \end{aligned}$$

Because this is a second order equation, and two initial conditions are given, this problem has a unique solution. In fact, this solution is $f(x) = 5\cos x - 7\sin x$ (you can verify that this satisfies both the differential equation and the two initial conditions). ◁

Solving an initial value problem almost always consists of two steps.

1. Find the general solution to the differential equation.
2. Plug in the initial conditions to the general solution. Solve for the constants.
3. Substitute the value you found for the constants to write down the solution to the initial value problem.

Example 1.9. In example 1.8, the general solution to the differential equation is $f(x) = A\cos x + B\sin x$ (we'll see how to find this general solution soon). Therefore

$$\begin{aligned} f(x) &= A\cos x + B\sin x \\ f'(x) &= -A\sin x + B\cos x \\ f(0) &= A\cos 0 + B\sin 0 \\ &= A \\ f'(0) &= -A\sin 0 + B\cos 0 \\ &= B \end{aligned}$$

This shows that $5 = f(0) = A$ and $-7 = f'(0) = B$, so $A = 5$ and $B = -7$ and $f(x) = 5\cos x - 7\sin x$ is the solution to the initial value problem. ◁

The fact that this set of initial conditions uniquely determines the function is the content of the Existence and Uniqueness Theorem for ordinary differential equations, which I will not state precisely here. This theorem explains why (as I mentioned in the previous section), once you find "enough" solutions to a differential equation (i.e. an expression with enough constants in it), you've found the general solution: if there are enough constants, then there will be enough flexibility to satisfy any particular initial conditions, hence enough flexibility to find all solutions (there are some technical details about this statement which I won't dwell on here).

2 Linear homogeneous differential equations

The first type of differential equation that we consider in this course are linear homogeneous differential equations. I have chosen to discuss these first because they are a model that is frequently used (albeit as a simplification) in many physical situations, yet they are simple enough that they are essentially completely understood (a rare situation for differential equations).

One basic idea underlies the solution to linear homogeneous differential equations: instead of looking for *all* solutions, instead make a (somewhat lucky) guess: that there will be solutions of the form $f(t) = e^{\lambda t}$ for some constant λ. This basic idea provides (with a little more work) the general solutions to these differential equations.

Linear homogeneous differential equations have forms like the following.

$$\begin{aligned} f'(t) + 3f(t) &= 0 \\ f'(t) - 7f(t) &= 0 \\ f''(t) + 3f'(t) + 16f(t) &= 0 \\ f''(t) + 4f'(t) - f(t) &= 0 \end{aligned}$$

More formally, a linear homogeneous differential equation is any differential equation of the following form.

$$f^{(n)}(x) + c_{n-1}f^{(n-1)}(x) + \cdots + c_2 f''(x) + c_1 f'(x) + c_0 = 0$$

where $c_0, c_1, \cdots, c_{n-1}$ are constants.

These differential equations have two very fortunate properties (it is useful to convince yourself that these two properties are true by thinking about what happens when you plug $Cf(t)$ or $f_1(t) + f_2(t)$ into a linear differential equation).

- If $f(t)$ is one solution, then so is $Cf(t)$ for any constant C.
- If $f_1(t), f_2(t)$ are any two solutions, then so is $f_1(t) + f_2(t)$.

The reason these two properties are so useful is that they mean that *a little good guesswork goes a long way*, because once you find one or two solutions, you can put them together to get many more solutions. To illustrate this, consider the following three examples.

Equation	one solution	another solution	General solution
(I) $f'' = 0$	$f(t) = 1$	$f(t) = t$	$f(t) = C_1 t + C_2$
(II) $f'' + f = 0$	$f(t) = \sin t$	$f(t) = \cos t$	$f(t) = C_1 \sin t + C_2 \cos t$
(III) $f'' + 3f' + 2f = 0$	$f(t) = e^{-t}$	$f(t) = e^{-2t}$	$f(t) = C_1 e^{-t} + C_2 e^{-2t}$

In all three cases, you might find the first two solutions by inspired guesswork. But once you do, you can immediately write down the general solution by multiplying the first two by arbitrary constants and adding them together. So you can solve the entire problem with just a couple good guesses (the number of good guesses should be equal to the order of the differential equation).

In fact, there is one particular way to make a “good guess” that gets you most of the way to the solution.

2.1 The characteristic equation

To make some inspired guesses at solutions to linear homogeneous equations, look for one particular kind of solution: solutions of the form $f(t) = e^{\lambda t}$, where λ is a constant that you will have to choose. These functions are very well behaved: the derivatives proceed as follows.

$$\begin{aligned} f(t) &= e^{\lambda t} \\ f'(t) &= \lambda e^{\lambda t} \\ f''(t) &= \lambda^2 e^{\lambda t} \\ &\cdots \\ f^{(n)}(t) &= \lambda^n e^{\lambda t} \end{aligned}$$

Therefore, if you want to see whether this particular function satisfies a linear differential equation, the problem simplifies as follows (shown in the case of a second order equation).

$$\begin{aligned} f''(t) + bf'(t) + cf(t) &= 0 \\ \lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} &= 0 \\ (\lambda^2 + b\lambda + c)e^{\lambda t} &= 0 \\ \lambda^2 + b\lambda + c &= 0 \end{aligned}$$

Therefore, to find *this particular type of solution*, it suffices to find values of λ satisfying an algebraic equation. This algebraic equation is called the *characteristic equation* of the differential equation. In the second order case, you can form it as follows.

Differential equation:	$f''(t) + bf'(t) + cf(t) = 0$
Characteristic equation:	$\lambda^2 + b\lambda + c = 0$

For higher order equations, higher derivatives simply lead to higher powers of λ.

In general, we have the following fact: *a number λ is a solution to the characteristic equation if and only if the function $e^{\lambda t}$ is a solution to the differential equation.* Therefore the characteristic equation gives a way to find solutions to the differential equation.

The behavior of the solutions to the differential equation depend on what kind of solutions occur to the characteristic equation. Consider what this looks like in the three main examples that we are considering.

Differential equation	Characteristic equation	Roots of characteristic equation
(I) $f'' = 0$	$\lambda^2 = 0$	$\lambda = 0$ (only one root)
(II) $f'' + f = 0$	$\lambda^2 + 1 = 0$	$\lambda = i, \lambda = -i$
(III) $f'' + 3f' + 2f = 0$	$\lambda^2 + 3\lambda + 2 = 0$	$\lambda = -1, \lambda = -2$

These three equations display the three cases which must be understood: real solutions, complex solutions, and repeated solutions.

2.2 Real roots

Consider first the situation in equation (III): the characteristic equation has only real roots. In fact, this case is the simplest: each of the two roots gives an exponential solution, and these two solutions together give the general solution.

Example 2.1. Consider the following differential equation.

$$f''(t) - 8f'(t) + 7f(t) = 0$$

The characteristic equation is $\lambda^2 - 8\lambda + 7 = 0$. This can be factored as $(\lambda - 1)(\lambda - 7) = 0$, hence there are two real roots: 1 and 7. Therefore there are two exponential solutions $f(t) = e^t$ and $f(t) = e^{7t}$ to this differential equation, and the general solution is $f(t) = C_1e^t + C_2e^{7t}$. ◁

2.3 Complex roots

Consider now what happens when the characteristic equation has complex roots (i.e. roots that are not real numbers). For example, the characteristic equation of equation (II) has roots i and $-i$.

Complex roots can still be used to obtain real solutions. The bridge from complex to real is *Euler's formula.*

$$e^{it} = \cos t + i \sin t$$

To get real solutions from complex solutions, use the following fact (it is useful to try to convince yourself this is true).

Fact. If $f(t) = g(t) + ih(t)$ is a complex solution to a linear differential equation (where $g(t)$ and $h(t)$ are real functions), then both $g(t)$ and $h(t)$ are real solutions to the original differential equation.

What this fact means is that a complex solution is actually a two-for-one solution: it has two parts (real and imaginary), each of which gives a solution to the differential equation.

Example 2.2. Consider the differential equation $f'' + 4f' + 13f = 0$.

The characteristic equation is $\lambda^2 + 4\lambda + 13 = 0$. By the quadratic formula, this equation has two complex roots: $-2 \pm 3i$. Now, observe that $e^{(-2+3i)t} = e^{-2t} \cdot e^{3it} = e^{-2t}\cos(3t) + ie^{-2t}\sin(3t)$. By considering the real and imaginary parts by themselves, we obtain two real solutions to the differential equation.

$$\begin{aligned} f(t) &= e^{-2t}\cos(3t) \\ f(t) &= e^{-2t}\sin(3t) \end{aligned}$$

Therefore, the general solutions is obtained from these two particular solutions.

$$f(t) = C_1e^{-2t}\cos(3t) + C_2e^{-2t}\sin(3t)$$

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This example illustrates the picture that we see in general: solutions to linear homogeneous differential equations arising from complex solutions of the characteristic equation are a sort of hybrid between exponential and sinusoidal functions. The exponential function determines a rate of decay (or explosion), while the sine and cosine factors determine an oscillation.

2.4 Repeated roots of the characteristic equation

There is one special case that needs different treatment: when the characteristic equation has a repeated root. The simplest example of this is equation (I), where the characteristic equation is $\lambda^2 = 0$, which has only one root $\lambda = 0$. This root does, in fact, give a solution $f(t) = e^{0t}$, also known as $f(t) = 1$. The other solution is, in fact, $f(t) = t$.

It turns out that the behavior of equation (I) really encapsulates what must always happen when the characteristic equation has a repeated root. This one root will give an exponential solution $f(t) = e^{\lambda t}$, and it turns out that another solution will always be $te^{\lambda t}$.

Example 2.3. Consider the differential equation $f'' - 2f' + f = 0$.

The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, which factors as $(\lambda - 1)^2 = 0$. So there is only one real root: $\lambda = 1$. So we obtain only one solution this way: $f(t) = e^t$. Because this root was repeated, another solution is $f(t) = te^t$.

Since e^t, te^t are two independent solutions, combining them gives the general solution.

$$f(t) = C_1e^t + C_2te^t$$

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