

n^{th} roots of complex numbers

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This note describes how to solve equations of the form $z^n = c$, where c is a complex number. These problems serve to illustrate the use of polar notation for complex numbers.

1 Polar and rectangular form

Any complex number can be written in two ways, called rectangular form and polar form.

$$\begin{aligned}\text{Rectangular: } z &= x + iy \\ \text{Polar: } z &= re^{i\theta}\end{aligned}$$

To convert from one form to the other, use *Euler's formula*.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

To convert from polar to rectangular: If you have a complex number in the form $re^{i\theta}$, it is relatively straightforward to convert it into the form $x + iy$.

Euler's formula says that $re^{i\theta}$ is the same as $r \cos \theta + ir \sin \theta$. This is $x + iy$, where

$$\begin{aligned}x &= r \cos \theta \\ y &= r \sin \theta\end{aligned}$$

These are the same equations that you always use to convert a pair (r, θ) of polar coordinates to rectangular coordinates (x, y) .

Example 1.1. Consider the complex number $4e^{-i\pi/6}$. It is currently expressed in polar form, with $r = 4$ and $\theta = -\pi/6$. Therefore in rectangular form, it can be expressed as

$$\begin{aligned}4e^{-i\pi/6} &= (4 \cos(-\pi/6) + i \cdot 4 \sin(-\pi/6)) \\ &= 4 \cdot \frac{\sqrt{3}}{2} + i \cdot 4 \cdot \left(-\frac{1}{2}\right) \\ &= 2\sqrt{3} - 2i\end{aligned}$$

The rectangular form is $2\sqrt{3} - 2i$. This is $x + iy$, where $x = 2\sqrt{3}$ and $y = -2$.

To convert from rectangular to polar: If you have a complex number in the form $x + iy$, you can convert it to the polar form $re^{i\theta}$ by finding suitable values of r and θ .

The radius r is easy: use the formula

$$r = \sqrt{x^2 + y^2}$$

The angle θ is a bit more finicky. You usually use the formula

$$\text{(almost) } \theta = \arctan \frac{y}{x}$$

but there are two caveats.

- If x is negative, add (or subtract) π to the result. This is because the arctan function always gives a value between $-\pi/2$ and $\pi/2$, so it will never give you ray pointing in the negative x direction.
- if $x = 0$, then y/x is not defined. In this case the point $x + iy$ lies on the y -axis (the imaginary axis). Then θ is either $\pi/2$ (if y is positive) or $-\pi/2$ (if y is negative).

These caveats are tricky at first, but they are not too hard to remember if you close your eyes and try to picture where your point is (i.e. which quadrant).

Example 1.2. Suppose that you wish to convert $-3 + 4i$ to polar form. So $x = -3$ and $y = 4$. You want to find the numbers r and θ . Finding r is easy:

$$r = \sqrt{(-3)^2 + 4^2} = 5$$

To find θ , begin with $\arctan(-4/3)$ (this is approximately -0.927 radians, or -53.13°). Since x is negative (the point is in the second quadrant), you should add or subtract π . I will add π :

$$\begin{aligned} \theta &= \arctan(-4/3) + \pi \\ &\approx 2.214 \text{ (or } 126.87^\circ) \end{aligned}$$

So the polar form of this complex number is $5e^{i(\arctan(-4/3)+\pi)}$, or approximately $5e^{2.214i}$.

Polar form and rectangular form are interchangeable. Any complex number can be written in one way or the other. You can use whichever form is most convenient for a particular problem. Generally speaking, polar form is easiest to use when you're doing multiplication, and rectangular form is easiest to use when you're doing addition.

2 The equation $z^n = 1$

The equation $z^n = 1$ has n complex solutions. These solutions are called *roots of unity* (but you don't need to know this term). The easiest way to write down these solutions is using the polar form for complex numbers.

2.1 How to think through the problem

This problem is about multiplying a number by itself many times, so polar coordinates are a natural choice.

Write $z = re^{i\theta}$. The numbers r and θ are what we want to find. We can also assume that r is positive and that θ is in $[0, 2\pi)$, since *any* complex number (except 0, which is not a solution) can be written in this way.

If z solves our equation $z^n = 1$, that means that

$$\begin{aligned} (re^{i\theta})^n &= 1 \\ r^n e^{i\theta \cdot n} &= 1 \end{aligned}$$

This means that $r^n e^{in\theta}$ must be a way to write the number 1 in polar form. In particular, we must have $r^n = 1$, since this has to be the absolute value of 1. So $r = 1$, since that's the only positive real number whose n th power is 1. So $e^{in\theta}$ must be a way to write 1 in polar form.

Now think about the angle θ . When θ is multiplied by n , the result must be the correct angle for the complex number 1. This angle is 0 radians. So $\theta = 0$ will give one solution: $z = e^{i \cdot n \cdot 0} = 1$.

Why isn't $\theta = 0$ the only possible solution? Because there are other angles θ so that $n\theta$ is the same *as an angle* as 0 (when I say "the same as an angle," what I mean is that it has the same sine and cosine, hence describes the same point in polar coordinates). It is also possible that

$$n\theta = 2\pi k$$

for any integer k . This gives the rest of the possible values of θ :

$$\begin{aligned}\theta &= 2\pi k/n \\ z &= e^{2\pi i k/n}\end{aligned}$$

Now there is a new puzzle: there appear to be infinitely many solutions! But in fact there are not: taking $k = 0$ and $k = n$ give the same one (the resulting values of θ differ by 2π). Since we only need to care about angles θ is $[0, 2\pi)$ (since any other angle is equivalent to one of these), we can restrict ourselves to only the values $k = 0, 1, 2, \dots, (n-1)$ such that $0 \leq 2\pi k/n < 2\pi$. This gives all the solutions to $z^n = 1$, expressed in polar form.

2.2 The solutions

The solutions can be written as follows.

$$\text{(in polar form)} \{1, e^{2\pi i/n}, e^{4\pi i/n}, e^{6\pi i/n}, \dots, e^{2(n-1)\pi i/n}\}$$

There is another suggestive way to write these solutions: if you let ω denote the second solution listed, that is $\omega = e^{2\pi i/n}$, these solutions are precisely $\{\omega^0, \omega^1, \omega^2, \dots, \omega^{n-1}\}$.

To express these solutions in rectangular form, use Euler's formula.

$$\begin{aligned}\text{(in rectangular form)} \quad & 1 (= \cos(0) + i \sin 0) \\ & \cos(2\pi/n) + i \sin(2\pi/n) \\ & \cos(4\pi/n) + i \sin(4\pi/n) \\ & \cos(6\pi/n) + i \sin(6\pi/n) \\ & \dots \\ & \cos(2(n-1)\pi/n) + i \sin(2(n-1)\pi/n)\end{aligned}$$

These n solutions are always evenly spaced around a circle of radius 1 in the complex plane.

Example 2.1. Consider the equation $z^6 = 1$. This has six solutions, as follows. Here I have simplified $2\pi/6$ to $\pi/3$ in the left column, and simply evaluated the cosine and sine of each angle in the right column.

Polar form	Rectangular form	
1	= 1	(because $\cos 0 = 1, \sin 0 = 0$)
$e^{i\pi/3}$	= $\frac{1}{2} + i\frac{\sqrt{3}}{2}$	(because $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$)
$e^{2i\pi/3}$	= $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	(because $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$)
$e^{3i\pi/3}$	= -1	(etc.)
$e^{4i\pi/3}$	= $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	
$e^{5i\pi/3}$	= $\frac{1}{2} - i\frac{\sqrt{3}}{2}$	

You could also more succinctly summarize the rectangular forms of these complex numbers as follows.

$$\pm 1 \text{ and } \left(\pm \frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right)$$

These six complex numbers are arranged in a hexagon around the origin.

3 The equation $z^n = c$ in general

To solve the equation $z^n = c$ for other complex numbers (besides $c = 1$), follow the same ideas as the previous section.

- Express c in polar notation, as $c = re^{i\theta}$.
- Notice that any z such that $z^n = c$ can be written in the polar form $\sqrt[n]{r}e^{i\phi}$, where ϕ is an angle so that $n\phi = \theta + 2\pi k$ for some integer k (this just means that $n\phi$ has the same sine and cosine as θ).

Example 3.1. Consider the equation $z^2 = 4i$. In other words, we are trying to find the “square root of i ” (scare quotes because there isn’t one square root, but two of them).

The number $4i$ has polar form $4e^{i\pi/2}$. Taking the square root of 4, we see that solutions to $z^2 = 4i$ must have the form

$$z = 2e^{i\phi}$$

where ϕ is an angle such that $2\phi = \pi/2 + 2\pi k$ for some k . In other words, $\phi = \pi/4 + \pi k$. We only have to care about two of these, because the rest are equivalent (differ by multiples of 2π): $\phi = \pi/4$ and $\phi = \pi/4 + \pi = 5\pi/4$.

So there are two solutions to $z^2 = 4i$. In polar form, they are

$$z = 2e^{i\pi/4}, 2e^{5\pi i/4}$$

Alternatively, these could be written in rectangular coordinates as

$$z = \pm(\sqrt{2} + i\sqrt{2})$$

(because $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$, and $\cos(5\pi/4) = \sin(5\pi/4) = -\sqrt{2}/2$).

3.1 A second method

There is another way to solve $z^n = c$ which you may (or may not) find easier.

- Express c in polar form as $re^{i\theta}$.
- Find *one* solution: $\sqrt[n]{r}e^{i\theta/n}$.
- Find the other solutions by solving $z^n = 1$, and multiplying your first solution by these n complex numbers.

Example 3.2. Consider the equation $z^4 = -16$. In polar notation, -16 is $16e^{\pi i}$. Therefore, just by taking the 4th root of 16 and dividing the angle by 4, we can find one solution right away: $z = 2e^{i\pi/4}$. In rectangular notation, this is $\sqrt{2} + i\sqrt{2}$.

Now, to find the other four solutions, solve the equation $z^4 = 1$. This has four solutions $1, i, -1, -i$ (you can find these as in the previous section). Therefore the four solutions to $z^4 = -16$ are

$$\begin{aligned}\sqrt{2} + i\sqrt{2} \\ i(\sqrt{2} + i\sqrt{2}) &= -\sqrt{2} + i\sqrt{2} \\ -1 \cdot (\sqrt{2} + i\sqrt{2}) &= -\sqrt{2} - i\sqrt{2} \\ -i \cdot (\sqrt{2} + i\sqrt{2}) &= \sqrt{2} - i\sqrt{2}\end{aligned}$$

These could also be summarized as $\pm\sqrt{2} \pm i\sqrt{2}$.