# Tropical Curves

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**Abstract**

A tropical curve is a graph with specified edge lengths, some of which may be infinite. Various facts and attributes about algebraic curves have analogs for tropical curves. In this article, we focus on divisors and linear series, and prove the Riemann-Roch formula for divisors on tropical curves. We describe two ways in which algebraic curves may be transformed into tropical curves: by aboemas and by specialization on arithmetic surfaces. We discuss how the study of linear series on tropical curves can be used to obtain results about linear series on algebraic curves, and summarize several recent applications.

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# 1 Introduction

The origins of tropical geometry lie in the study of tropical algebra, whose basic object is the set $\mathbf{R} \cup \{-\infty\}$ equipped with the operations $x \oplus y = \max(x, y)$ and $x \otimes y = x + y$. This is the so-called *tropical semifield*. Tropical algebra may be regarded as an idealization of ordinary algebra composed with the logarithm map, insofar as $\log(x \cdot y) = \log x \otimes \log y$ and $\log(x + y) \approx \log x \oplus \log y$ for values of $x$ and $y$ that are sufficiently far apart. Due to the fact that $x \oplus x = x$, the term "idempotent algebra" is sometimes used. Such semirings were originally studied by Imre Simon of Brazil. The appellation "tropical" arose apparently because French computer scientists believed that it would flatter Mr. Simon.

Although tropical mathematics in a broad sense has been studied for over two decades, it has recently seen enormous growth in its connections to algebraic geometry, and especially algebraic curves. Polynomials over the tropical semiring are piecewise-linear functions with integer slopes, hence their graphs form convex polyhedral complexes. These complexes bear many strong analogies to algebraic varieties. Along the lines of the tropical-logarithmic analogy made above, tropical geometry studies what happens to algebraic geometry after catastrophic deformations of complex structure. In order to make such analogies more precise, the notion of a *tropical variety* has recently been introduced; the notion of a *tropical curve* is somewhat older. An indication of the recent growth of the field is that as recently as 2005, there was no satisfactory definition of tropical varieties in all dimensions ([11]).

This article is meant as an introduction to some basic topics about tropical curves, although we sometimes indicate how the situation should be generalized to higher-dimensional tropical varieties. We do this in order to maintain simplicity of notation and to preserve as much geometric intuition as possible. We will focus on the notion of divisors and linear series on tropical curves. Our approach in this article is to avoid, as much as possible, discussion of the tropical semifield and tropical algebra, and to instead phrase our statements in terms of piecewise-linear maps. We hope that this will minimize the confusion associated between keeping the distinction between classical and tropical arithmetic straight, while still allowing us to cover nontrivial content and applications.

We shall examine two methods to converting curves to tropical curves: amoebas and specialization via arithmetic surfaces. Amoebas are the result of applying the logarithm map to all coordinates of an algebraic variety, and then deforming the variety to shrink the image onto a polyhedral skeleton; we indicate how this can be understood and formalized for curves. Specialization uses techniques from scheme theory to degenerate a smooth curve into a graph. In section 2 we describe these two methods informally. In sections 5 and 6, we elaborate on these two methods, and discussion a recent application of each to classical geometry: section 5.6 discusses the work of Mikhalkin [19] on the computation via tropical geometry of Gromov-Witten and Welschinger invariants, and section 6.3 discusses a recent tropical proof of the classical Brill-Noether theorem, by Cools, Draisma, Payne, and Robeva [8]. Sections 3 and 4 formulate the notions of abstract tropical curves and linear series on them, so that we may draw on these notions in the following two sections. The capstone of section 4 is the tropical Riemann-Roch theorem, which is a compelling testament to the analogy between the tropical and classical worlds. We also believe that its proof sheds light on the principles behind the classical Riemann-Roch theorem.

An excellent survey of tropical curves (although now outdated) is [11]. For definitions of tropical varieties in arbitrary dimension, as well as a notion of tropical schemes, see Mikhalkin's paper [21] and book in progress [20]. An illuminating, if rather technical, introduction to tropical geometry via toric schemes, with particular attention to tropical intersection theory, is [16]. A delightful paper

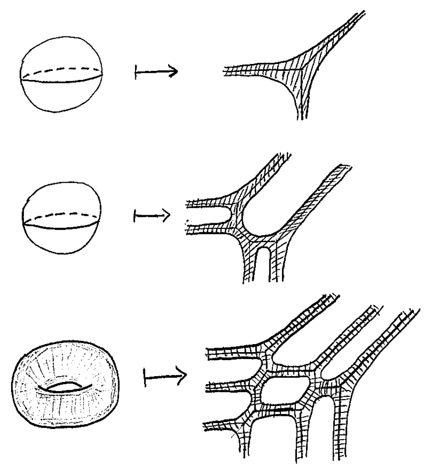

Figure 1: Plane curves of degrees 1, 2 and 3 under the Log map.

that highlights the analogy between Riemann surfaces and finite graphs, with many charming combinatorial implications, is [4]. For more careful discussion of the polyhedral aspects of the subject, computational aspects and the development of so-called tropical grassmannians, consult [25], [26], [27], or [28]. Other references to topics specific to individual sections are distributed throughout this article.

# 2 From curves to graphs

Before defining abstract tropical curves in section 3, we describe how graphs arise from degeneration of curves, in order to motivate which features must be brought out in tropical curves, and which properties of algebraic curves we might expect to carry over to them.

## 2.1 Amoebas of plane curves

We begin informally by describing in vague terms some intuition behind the definitions of tropical geometry. We will become more precise in the next subsection, where we describe a suitable formalism for the intuitive content of this subsection.

Suppose that $C$ is a general algebraic curve given by a degree $d$ polynomial in $(\mathbf{C}^*)^2$ (or in any compactification thereof). Then $C$ may be mapped into $\mathbf{R}^2$ by applying the map $Log : (x, y) \mapsto (\log|x|, \log|y|)$. This map splatters the curve onto the plane; the result is called the *amoeba* of the curve. Figure 2.1 illustrates the effect of the Log map on plane curves of degree 1, 2, and 3.

Several features of these amoebas are immediately apparent. First, they appear to have the same number of holes as the original curve. Second, far away from the origin, the amoebas consist of three sets of long, straight tentacles: one set pointing in each of the directions west, south, and northeast, and each set containing $d$ tentacles (where $d$ is the degree of the curve). As the reader can verify, these should be expected to exist as the manifestation of points where one of the coordinates approaches either 0 or $\infty$. Third, near the center of the picture, the amoeba seems to cling to a skeleton of sorts. Our objective is to uncover this skeleton, which we shall refer to as a *tropical curve*, and understand how properties of the original curve correspond to properties of the skeleton.

To achieve this, we consider not just a single curve, but a family of curves parameterized by $t$, which will shrink to 0. We could like to choose this family in such a way that the amoeba is neatly shrinkwrapped onto the underlying skeleton as $t$ goes to 0. The following section indicates an effective way to formalize this idea.

## 2.2 Curves over the field of Puiseux series

Suppose that $p_t(x, y)$ is a family of polynomials in $x, y$, for $t$ in some neighborhood of 0. Then the coefficients of $p_t$ are analytic functions of $t$, i.e. convergent power series. Disregarding the convergence of this power series, we have simply a curve defined over the ring $\mathbf{C}[[t]]$. We wish to understand the image of this curve under the Log map, and how it specializes as $t$ goes to 0. The following lemma is very useful.

**Lemma 2.1.** *Let $R = \mathbf{C}[[t]]$ and $K = \mathbf{C}((t))$ be the field of fractions of $R$. Then every field extension $K'$ of $K$ is obtained by adjoining a $k^{th}$ root of $t$ for some $k$, and the integral closure of $R$ in such an extension is $R' = \mathbf{C}[[t^{1/k}]]$.*

*Proof.* See [10], corollary 13.15. □

This suggests the following definition.

**Definition 2.2.** The *field of Puiseux series*, which we denote $\mathbf{K}$, is given by

$$\mathbf{K} = \bigcup_{k \geq 1} \mathbf{C}((t^{1/k})),$$

where the union is taken in an algebraic closure of $\mathbf{C}((t))$. For any $x \in \mathbf{K}^*$, we denote by $\mathrm{val}(x)$ the smallest exponent of $t$ occurring in $x$.

By lemma 2.1, the field $\mathbf{K}$ is in fact algebraically closed. The map val is well-defined since all the exponents of terms in a given series $x \in \mathbf{K}$ have bounded denominator and are bounded below. Because the field $\mathbf{K}$ is algebraically closed and has characteristic zero, theorems over $\mathbf{K}$ almost always suffice to prove theorems over $\mathbf{C}$.

Returning to our original situation: a curve defined over the ring $\mathbf{C}[[t]]$ gives rise to a curve over the algebraically closed field $\mathbf{K}$. Suppose that this curve lies in $\mathbf{A}^2_{\mathbf{K}}$, and is given by coordinates $(x(t), y(t))$, where $x, y$ are Puiseux series. Then supposing for then moment that both $x$ and $y$ are convergent power series, consider their image under the Log map. As $t$ goes to 0, the result will increasingly be dominated by the term of lowest exponent; hence the result will be approximately $(\mathrm{val}(x) \log |t|, \mathrm{val}(y) \log |t|)$. As $t$ approaches 0, $\log |t|$ will become a very large negative number. Thus we should scale by $(-\log |t|)$ in order to prevent this expansion. Alternatively, we should take the logarithm with respect to $1/|t|$. This now suggests that the following definition as the correct abstract notion for "the limit of amoebas."

**Definition 2.3.** Let $C$ be a curve in $(\mathbf{K}^*)^2$. The *non-archimedean amoeba* of $C$ is the closure in $\mathbf{R}^2$ of the image of $C$ under the map $(x, y) \mapsto (-\mathrm{val}(x), -\mathrm{val}(y))$.

Observe that if $p(x, y)$ is a polynomial with coefficients in $\mathbf{K}$, then if it is equal to 0, then two of its monomials must be equal, and both have minimal valuation among all the monomials (otherwise, the valuation of $p(x, y)$ would be equal to the lowest valuation of a monomial). In fact, we have the following theorem, originally proved by Kapranov. A proof can be found in [9].

**Theorem 2.4.** *Let $p(x, y) = \sum_{i,j} c_{i,j} x^i y^j$ be a polynomial, defining a curve $C$ in $(\boldsymbol{K}^*)^2$. Then the non-archimedean amoeba of $C$ is precisely the closure of the set of points $(u, v) \in \boldsymbol{R}^2$ where $\max\{-val(c_{i,j}) + iu + jv\}$ is achieved more than once; in other words, it is the set of points where this function is not differentiable.*

One inclusion follows from the remarks before the theorem statement. The other follows from a suitable application of Hensel's lemma, although we do not wish to dwell on the details here. See section 5.5 for a related argument that suggests the proof.

The non-archimedean amoeba of a plane curve defined over $\mathbf{K}$ gives one way to associate a graph, in this case embedded in $\mathbf{R}^2$, to such a curve. We now consider another construction, which considers not the limit of the amoeba as $t$ goes to 0, but instead the limit of the curve. This notion does not work for all curves $C$, but can still be used to prove nontrivial theorems for curves defined over algebraically closed fields of any characteristic, one example of which we will describe in section 6.

### 2.3 Specialization

A second method for converting curves into graphs, discussed by Baker in [2] and applied to algebraic curves over fields of any characteristic in [8], is by specialization on an arithmetic surface. Although this method is different in nature from the amoeba approach of the previous subsection, there is a deep connection between the two methods of converting curves to graphs, which is elaborated in [24].

This method is based on the idea of specialization, which is described in section II.8 of [20]. This section, and section 6, which builds upon it, depend on various aspects of scheme theory that will not be used elsewhere in this article.

As in the previous section, suppose that $C$ is a curve defined over the field $\mathbf{K}$ of Puiseux series. Such a curve can be defined over some extension of the field of formal Laurent series, and by lemma 2.1, it suffices to take this extension to be $\mathbf{C}((t))$ itself, by replacing $t^{1/k}$ by $t$. Since this is the field of fractions of the complete discrete valuation ring $\mathbf{C}[[t]]$, we can study such curves in terms of arithmetic surfaces.

**Definition 2.5.** Let $R$ be a complete discrete valuation ring, with field of fractions $K$ and algebraically closed residue field $k$. An *arithmetic surface* is a proper flat scheme $\mathfrak{X}$ over Spec$R$, whose generic fiber is a smooth curve $C$ over $K$. Such an arithmetic surface is called a *model* for $C$.

For a given curve $C$, there may exist more than one model for $C$. If $C$ is meant to describe a family of curves, then the limit as $t$ goes to 0 should be precisely the special fiber $\mathfrak{X}_k$ of this family, which will be a one-dimensional scheme over $k$. There are several convenient attributes the special fiber may have.

**Definition 2.6.** A model $\mathfrak{X}$ for a curve $C$ is called *semistable* if the special fiber $\mathfrak{X}_k$ is a reduced scheme over $k$ with only simple double points as singularities. The model $\mathfrak{X}$ is called *strongly semistable* if each irreducible component of $\mathfrak{X}_k$ is smooth, and $\mathfrak{X}$ is called *totally degenerate* if each irreducible component of $\mathfrak{X}_k$ is a genus 0 curve.

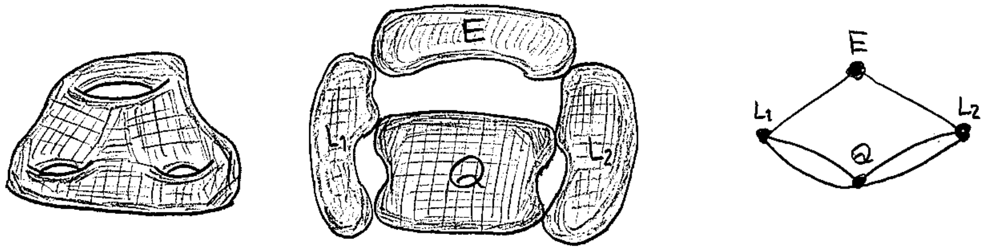


Figure 2: A smooth plane quartic curve $X$ (over $\mathbf{K}$), its special fiber $\mathfrak{X}_k$ (over $\mathbf{C}$), and the corresponding dual graph $G$.

A semistable model $\mathfrak{X}$ for the curve $C$ gives rise to a graph, as follows. Notice that the model is strongly semistable if and only if the resulting graph has no loop edges.

**Definition 2.7.** The *dual graph* of the special fiber of a semistable model $\mathfrak{X}$ for $C$ is a multigraph whose vertices correspond to the irreducible components of $\mathfrak{X}_k$, and whose edges correspond to the double points of $\mathfrak{X}_k$ and connect the two components involved.

*Example* 2.8 (Adapted from [2]). Let R be the ring $\mathbf{C}[[t]]$. Consider the arithmetic surface $\mathfrak{X}'$ in $\mathbf{P}^2_R$ given by the homogenous quartic polynomial $p(X,Y,Z) = (X^2 - 2Y^2 + Z^2)(X^2 - Z^2) + tY^3 Z$. The generic fiber may be regarded as a smooth quartic plane curve $C$ over $\mathbf{K}$. By the genus formula, $C$ has genus $\binom{4-1}{2} = 3$. This scheme is not regular, but blowing up the point $[0,1,0]$ in the special fiber results in an arithmetic surface $\mathfrak{X}$ with generic fiber $C$. The special fiber $\mathfrak{X}_k$ consists of a plane conic $Q$ meeting each of two lines $L_1, L_2$ in two points, and the exceptional divisor, a genus 0 curve meeting each the lines $L_1, L_2$ in one point each. The result is shown in figure 2, along with the dual graph, $G$. Observe that $G$ has 3 loops; the genus of $G$ (as defined in section 3) is equal to the genus of $C$. We will see in section 6.2 that this is always the case for a totally degenerate strongly semistable model.

Point on the curve $C$ can be associated to points on the special fiber $\mathfrak{X}_k$ via the specialization map, which will be discussed in section 6. If $\mathfrak{X}$ sits in $\mathbf{P}^n_R$, then this map is the specialization map described in [20].

One strength of the specialization approach is that it can be used to prove results that are valid in all characteristics, as we will see is section 6.

The specialization approach is particularly well-suited to problems concerning the dimensions of linear series on the curve $C$. Linear series on $C$ specialize to linear series on the graph $G$, and ranks of linear series do not decrease under specialization (this is the content of the specialization lemma 6.1). Before describing these topics, we must describe abstract tropical curves and linear series on them, to provide the necessary vocabulary and technique.

## 3 Metric graphs and tropical curves

Just as smooth manifolds are topological spaces composed of euclidean neighborhoods glued by smooth maps, tropical varieties are topological spaces composed of neighborhoods that appear as polyhedral cones, glued by continuous piecewise affine linear maps with integer slope. The requirement that the maps have integer slopes (rather than simply being piecewise linear) is critical,

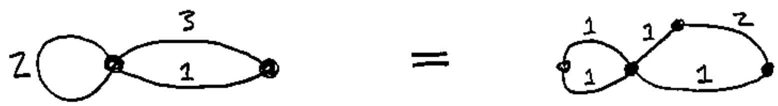


Figure 3: Subdivision is not regarded as changing the metric graph $\Gamma$.

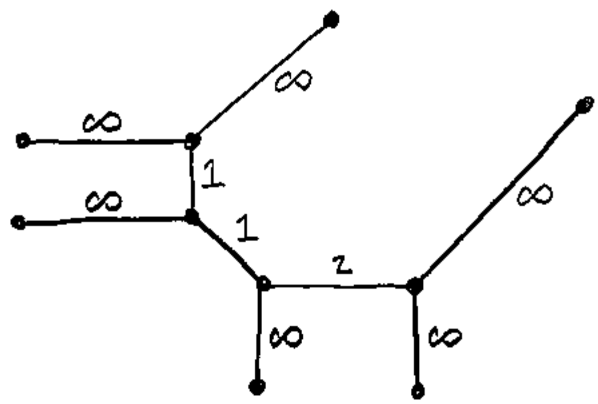


Figure 4: A tropical curve.

and derives from the fact that we are meant to regard maps of tropical varieties as the image under a logarithmic map of maps of algebraic varieties. Monomials, naturally, have integer exponents. In this article, we are not concerned with general tropical varieties, but will instead only develop the case of tropical curves. We begin with an informal discussion before presenting a more careful definition.

Abstract tropical curves are a mild generalization of metric graphs. Intuitively, a metric graph is simply a finite graph together with edge lengths in $\mathbf{R}_+$. Two such graphs are regarded as the same if they give the same metric space; for example, splitting an edge into two edges whose lengths sum to the original length is not considered to change the metric graph.

A metric graph can be specified by a finite multigraph, possibly with loop edges, with assigned edge lengths. However, it is never necessary to include loops or multiple edges, since these can always be subdivided to produce a simple graph (see figure 3).

A tropical curve differs from a metric graph in that it allows the presence of infinite edges, so long as they lead to leaves (see figure 4). Some care is needed here in order to carefully define an infinite edge: the author prefers to conceive of an infinite edge as the one-point compactification of the interval $[0, \infty)$, where we regard this not just as a topological space but as almost a metric space (where the distance from any point to $\infty$ is infinite). Although this has the same topology as $[0, 1]$, the metric structure is different, insofar as there is a point with no metric neighborhoods. Thus infinite edges are asymmetric; it is the end at infinity that we require to be a leaf of the graph.

Several remarks are in order on the principle behind this definition. We wish to allow infinite edges on tropical curves in order to describe the behavior of the tentacles described in section 2.1, but clearly there can be no more combinatorial structure at the "other end" of such a tentacle. We might consider representing these edges as open intervals, however compactifying the space by adjoining vertices at infinity is convenient for a number of reasons. It constrains the definition of "piecewise linear" functions (which shall become important momentarily) to those with finitely many pieces. More importantly for our purposes, we shall see in section 4 that this convention allows the divisor of a rational function to be supported in part at infinity, which is necessary in order for the degree of this divisor to be zero. As for the fundamental asymmetry of infinite edges, this simply arises from the fact that in tropical geometry (in contrast to classical projective geometry), points at infinity are qualitatively different from finite points.

We shall now present a more precise definition of the notions above, along with a notion of a rational and regular functions on a tropical curve.

**Definition 3.1.** A *tropical edge* $E$ is a topological space together with an identifictation (i.e. a homeomorphism) with $[0, \ell]$, for some $\ell \in (0, \infty]$. A *finite point* on $E$ is any point other than $\infty$. The *vertices* of $E$ are 0 and $\ell$.

**Definition 3.2.** A *tropical curve* $\Gamma$ is a connected finite graph, together with an identification of each edge with a tropical edge, such that all infinite points are leaves. If $\Gamma$ has only finite points, it is called a *metric graph*. Two such labelled graphs are considered to give the same tropical curve if they have the same infinite points and their sets of finite points give the same metric space.

Note that we have chosen not to refer to two curves with a common subdivision as "equivalent", since elsewhere in the literature this term is reserved for a weaker notion in which all simply connected tropical curves are considered equivalent (for example, in [21]).

**Definition 3.3.** A *rational function* on a tropical edge $E$ is a piecewise linear function from $E$ to $\mathbf{R} \cup \{\pm\infty\}$, where all finite points are sent to finite values, and the slope of the function on all pieces is an integer. A *rational function* on a tropical curve $\Gamma$ is a continuous function from $\Gamma$ to $\mathbf{R} \cup \{\pm\infty\}$ which restricts to each edge as a rational function on that edge.

**Definition 3.4.** The *order* of a rational function $f$ at a point $p$ on a tropical curve $\Gamma$ is the sum of outgoing slopes of $f$ around $p$. The function $f$ is called *regular* at $p$ if it has nonnegative order at $p$. A *regular function* on $\Gamma$ is a rational function regular at all points in its domain.

Recall that we wish to regard tropical curves as being limits of images of curves under the Log map. Given this, we should expect the logarithm of a regular or rational function to come to resemble a function build up from the tropical operations $\oplus$ and $\otimes$; that is, a continuous piecewise linear function. This is the reasoning behind this notion of rational function. The notion of the divisor of a rational function similarly follows from reasoning about the limits of amoebas. Although we do not wish to go into tropical algebra in this section, we simply mention that the rational functions on a tropical curve do not form a field, but instead they form a semifield (with the operations $\oplus$ and $\ominus$). In fact, this semifield is also a $\mathbf{T}$-module, just as the ring of regular functions and field of rational functions on a variety are $\mathbf{C}$-modules. In [19], the theory of tropical varieties is developed using the idea of a sheaf of tropical algebras; one can similarly define sheaves of tropical modules and so forth. We shall not discuss any of that in this article.

*Example* 3.5. Consider the rational function $f(t) = \max(t - 2, 0) + \min(t, 1)$ on the tropical edge $[0, \infty]$. The points 0 and 2 have order 1, the points 1 and $\infty$ have order $-1$, and all other points have order 0. Notice in particular that the sum of the orders of all points is 0.

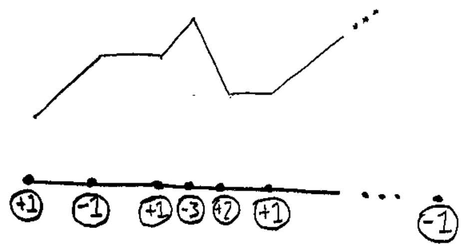


Figure 5: A rational function on an infinite tropical edge. Coefficients of its divisors are indicated.

Observe that a rational function can have nonzero order at only finitely many points of $\Gamma$. Hence rational functions can be used to define divisors on tropical curves, as we do in the following section.

Most applications of tropical curves to algebraic curves concern constructions in which the genus of the curve is the same as the genus of the tropical curve, in the following sense. This is true when the tropical curve is constructed as the limit of amoebas, or through specialization in a totally degenerate strongly semistable model.

**Definition 3.6.** The *genus* of a tropical curve is $\dim H_1(\Gamma)$, where $\Gamma$ is regarded as a topological space.

Notice that if $\Gamma$ is described by a multigraph, regardless of how edges are subdivided, the genus is $|E| - |V| + 1$.

We do not wish to delve too much into more general tropical varieties in this article, but we shall briefly remark on how such definitions must work. The reader can presumably imagine how the definitions above could be stated in terms of an atlas: a tropical curve is a topological space with some atlas of open sets that are all homeomorphic to "stars" (single vertices with some number of open rays emanating) such that the transition functions have integer slope. The reader may consult [21] for a general definition, along these lines, of tropical varieties of all dimensions; one must replace "stars" with "polyhedral cones" and "slope" must be understood to be the full derivative matrix while must lie in $GL_n(\mathbf{Z})$.

# 4 Divisors and linear equivalence on tropical curves

In this section, we shall discuss the notion of linear series on tropical curves, in analogy with the same notions on algebraic curves. These notions will provide the necessary language to state and prove the tropical Bézout theorem in section 5.14 and will be used in section 6, through the use of the specialization lemma, to obtain results about the corresponding notions on algebraic curves.

The definitions regarding linear series on tropical curves are completely analogous to the corresponding definitions for algebraic curves. After stating these definitions, we shall discuss the Riemann-Roch theorem for tropical curves.

All these definitions (and the Riemann-Roch theorem) were originally stated and proved by Baker and Norine [4] on graphs. These results were subsequently adapted to metric graphs and tropical curves in [22] and [12]. Baker and Norine also considered various aspects of the Jacobian of a graph, although much of their analysis does not carry over to tropical curves. Mikhalkin and Zharkov [22] have developed a more detailed theory of the Jacobian of a tropical curve, including tropical analogs of Abel's theorem and Jacobi inversion; their proof of Riemann-Roch is based on tropical Abel-Jacobi theory. We do not discuss these developments here, because they require a discussion of differentials and line bundles on tropical curves, which we have not developed.

**Definition 4.1.** Let $\Gamma$ be a tropical curve. The group of divisors $\mathrm{Div}(\Gamma)$ is the free abelian group generated by points of $\Gamma$. The *degree* of a divisor is the sum of its coefficients.

**Definition 4.2.** The *divisor of a rational function* $f$ on $\Gamma$, denoted $(f)$, is $(f) = \sum_{p\in\Gamma} \mathrm{ord}_p(f)$. Such divisors form a subgroup of $\mathrm{Div}(\Gamma)$, which is called the subgroup of *principle divisors.* Two divisors are called *linearly equivalent* if they differ by a principle divisor. The group $\mathrm{Pic}(\Gamma)$ is the quotient of the group of divisors by the group of principle divisors, i.e. the group of linear equivalence classes.

Observe that given a rational function $f$, each interval in $\Gamma$ on which $f$ is linear contributes some integer $m$ to the order of $f$ at one endpoint, and contributes $-m$ to the order of $f$ at the other endpoint. The following proposition is immediate. Observe that this justifies our choice of including infinite points in $\Gamma$, rather than leaving infinite edges half-open.

**Proposition 4.3.** *The degree of a principle divisor is* 0*, hence the degree of a divisor class in* $Pic(\Gamma)$ *is well-defined.*

As in the case of algebraic curves, we shall denote by $\mathrm{Pic}_n(\Gamma)$ the set of divisor classes of degree $n$.

We define one divisor of particular importance on tropical curves (in particular, for the statement of the Riemann-Roch theorem).

**Definition 4.4.** The *canonical divisor* of a tropical curve $\Gamma$ is given by

$$K = \sum_{p\in\Gamma} (\mathrm{Val}(p) - 2)p$$

Where $\mathrm{Val}(p)$, the valence of $p$, is the number of edges leaving $p$ (which is 2 at all but finitely many points).

The canonical divisor is intriguing partially because it is seemingly the only intrinsically defined divisor on $\Gamma$, which does not depend on a choice of subdivision of edges. Of course, it is far more intriguing because it is a dualizing divisor for Riemann-Roch purposes (section 4.2). For a discussion of why this definition should correspond to a canonical divisor on an algebraic curve, and more particularly why we have defined *one specific* divisor, instead of a class, see section 6.2.

The structure of $\mathrm{Pic}(\Gamma)$ on tropical curves is closely related to its structure on algebraic curves, as will be seen clearly in the next two subsections. The following two examples show the correspondence in genus 0 and 1.

*Example* 4.5. Suppose that $\Gamma$ has genus 0. Then $\mathrm{Pic}(\Gamma) \cong \mathbf{Z}$, via the degree map. To see this, observe that for any two finite points $p, q \in \Gamma$, there is a rational function given by $f(x) = d(p, x)$ on the unique path from $p$ to $q$, and constant elsewhere (namely, it is 0 on the component of $p$

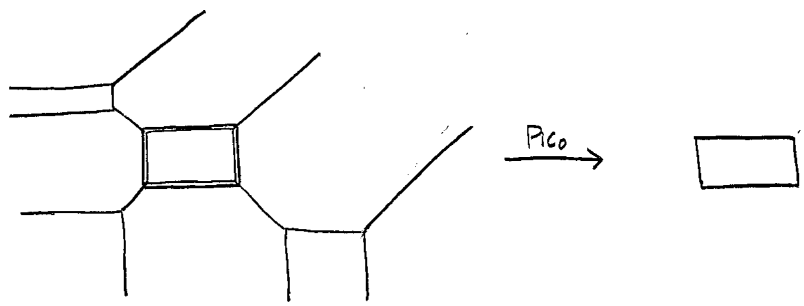


Figure 6: The Picard group of a genus 1 tropical curve is a circle, and can be identified with a nontrivial cycle in $\Gamma$. All points on attached trees are equivalent to the nearest point on this cycle.

and $d(p,q)$ on the component of $q$ on the space obtained by removing the path from $p$ to $q$). This gives $(f) = p - q$, hence $p$ and $q$ are linearly equivalent as divisors. As for infinite points, if $p$,$q$ are correspond to $0, \infty$ on an infinite edge, then the function given by $f(x) = x$ on this edge and $f(x) = 0$ elsewhere gives $(f) = p - q$. Thus every infinite point is equivalent to a finite point, and in turn to every other point on $\Gamma$. Thus linear equivalence is entirely given by degree.

Observe that the argument above also shows that if $\Gamma$ contains a tree that joins the rest of $\Gamma$ at a single point, then all points on this tree are linearly equivalent to the point where the tree joins the rest of the graph. For this reason, other authors consider $\Gamma$ to be equivalent as a tropical curve to the result of pruning off all such trees, as in [21]. Operations that remove such trees are called *contractions.*

*Example* 4.6 (The group law on tropical genus 1 curves). Suppose that $\Gamma$ has genus 1. Let $\Gamma'$ be the curve given by a subgraph corresponding to a nontrivial cycle. Then all points on $\Gamma \backslash \Gamma'$ are equivalent to a point on $\Gamma'$, by the discussion above (see figure 4.6). Hence $\text{Pic}(\Gamma) \cong \text{Pic}(\Gamma')$, so let us focus our attention on the cycle, $\Gamma'$. Let $\ell$ be the length of this cycle. We will show that $\text{Pic}_0(\Gamma) \cong \mathbf{R}/\ell$. The proof is entirely analogous to the argument that a genus 1 algebraic curve has Picard group isomorphic to a curve itself.

First, observe that if $p, q$ are two distinct points on $\Gamma'$, then they are not linearly equivalent as divisors: if they were, there would be a rational function $f$ such that $p - q = (f)$. Such a function would have slope $n$ on one path from $p$ to $q$, whose length we denote by $a$, and slope $1 - n$ on the other, whose length is $\ell - a$. But then $f(q) - f(p)$ would be both $a\lambda$ and $(\ell - a)(1 - \lambda)$, which implies that $\ell = \ell\lambda + a$, and hence that $a$ is an integer multiple of $\ell$, which is impossible. Hence no two points on $\Gamma'$ are linearly equivalent.

Next, we demonstrate that a group law holds on $\Gamma'$. Fix a base point $p_0 \in \Gamma'$, and identify $\Gamma'$ with $\mathbf{R}/\ell$ via distance from $a$ (which is well-defined modulo $\ell$). Suppose that $x, y, z$ are three values in $\mathbf{R}$ whose sum is a multiple of $\ell$, and let $p, q, r$ be the three corresponding points on $\Gamma'$. Assume without loss of generality that $0 \leq x \leq y \leq z < \ell$. Then let $f$ be the piecewise continuous function on $[0, \ell]$ with $f(0) = 0$, slope 0 on $(0, x)$, slope 1 on $(x, y)$, slope 2 on $(y, z)$ and slope 3 on $(z, \ell)$. Then $f(\ell) = 3\ell - x - y - z$, which is $k\ell$ for some integer $k$, by assumption of $x, y, z$. Hence letting $g(x) = f(x) - kx$, we obtain a well-defined rational function on $\Gamma'$ whose divisor is $(g) = p + q + r - 3p_0$. Now, define a map $\phi : \mathbf{R}/\ell \to \text{Pic}_0(\Gamma)$ by $\phi(x) = p - p_0$, where $p$ is the point distance $x$ from $p_0$. Then it follows from the linear equivalence described above that $\phi(x+y) = \phi(x)+\phi(y)$. Also, $\phi(0) = 0$, so $\phi$ is a homomorphism. The map $\phi$ is surjective, since its image contains all divisors $p - p_0$ and is closed under sum. The map is injective since $\phi(p) = \phi(q)$ if and only if $p$ is linearly equivalent to $q$, which is the case if and only if $p = q$ by the previous paragraph. Hence $\phi$ is an isomorphism.

Thus we have established that $\mathrm{Pic}_0(\Gamma)$ is isomorphic to a circle, with the usual group structure. Furthermore, this circle can be identified with any nontrivial cycle in $\Gamma$ in a natural way.

Mikhalkin defines moduli spaces of tropical curves in [21] (up to an equivalence relation that contracts trees). The moduli space of genus 1 curves is simply $\mathbf{R}_+$, where the parameter is the length $\ell$ above. In this sense, the length $\ell$ can be regarded as the tropical analog of the $j$-invariant of an elliptic curve.

Proceeding with the analogy with algebraic curves, we now define ranks of divisors on tropical curves. We would like, of course, to define this in terms of the dimension of the vector space of global sections of some line bundle. It is possible to define line bundles on tropical curves and to associate them to divisors (see [22]), but the sections of these line bundles do not form a vector space (although they do form a tropical module).

The established convention is to carry over a different, equivalent, description of ranks of divisors on algebraic curves. On an algebraic curve, a divisor $A$ has rank at least $n$ if and and if for all effective divisors of degree $n$, $|A-E|$ is nonempty. To see this, observe that for a section of the line bundle $\mathcal{L}(A)$ to vanish on an effective divisor $E$ is $\deg(E)$ linear conditions, hence if $r(A) \geq n$, then some nonzero section must vanish on $E$. Conversely, if $r(A) < n$, one can construct an effective divisor of degree at most $n$ such that $|A - E| = \emptyset$ inductively as follows: select any point $p$ that is not a base point for $|A|$ (i.e. such that not all global sections of $\mathcal{L}(A)$ vanish at $p$). Then $r(A - p) < r(A)$. Proceeding in this way will construct the desired divisor $E$.

Bearing in mind this equivalent formulation for algebraic curves, we make the following definition.

**Definition 4.7.** A divisor $E$ is *effective* if all its coefficients are nonnegative. The notation $A \leq B$ will indicate that $B-A$ is effective, and $|A|$ will denote the set of effective divisors linearly equivalent to $A$. The *rank* of a divisor $A$ is:

$$r(A) = \min\{\deg(E) : E \geq 0,\ |A - E| = \emptyset\} - 1. \tag{1}$$

Do not take the analogy between ranks of divisors on tropical and algebraic curves too literally; while a linear series on an algebraic curve is a bona fide geometric object whose dimension is the rank, the same is not true for linear series on tropical curves. Linear series on tropical curves can be understood geometrically as polyhedral complexes, as in [14], but the dimensions of these complexes are not the same as the rank.

The reader may well protest, on the basis of these pathologies, that we have surely defined ranks of divisors incorrectly, and all consideration of them is a sham. In our defence, we offer the specialization lemma 6.1, which is a powerful testament to the ability of our notion of rank to prove nontrivial facts about divisors on algebraic curves. We also offer the Riemann-Roch theorem, whose persistence under this definition of rank is surely a testament to its rich underlying content.

## 4.1 The Riemann-Roch criterion

Baker and Norine [4] prove that the Riemann-Roch formula can be deduced from an abstract criterion which only concerns divisors of degree $g - 1$. To motivate this criterion, observe that on an algebraic curve, if $D \leq E$, then $|E| = \emptyset$ implies that $|D| = \emptyset$. Hence, to understand which divisors give an empty linear series, it suffices to understand the maximal such divisors, under the ordering given by $\leq$. It can be deduced from the Riemann-Roch theorem and formula 1 that all such divisors on a curve have degree $g-1$. In addition, the set $\mathcal{N}$ of degree $g-1$ divisors with empty linear series is symmetric about any canonical divisor, in the sense that $K - \mathcal{N} = \mathcal{N}$. The basic observation in [4] is that the full Riemann-Roch formula can be recovered from only these two facts. Thus the tropical Riemann-Roch formula will follow from an understanding of the maximal divisors with empty linear series on graphs.

In order to formulate the Baker-Norine criterion in its full generality, we shall define the notion of a *divisor system*. We include this only for the sake of generality; the reader may simply think of divisors and divisor classes on an algebraic curve, graph, or tropical curve.

**Definition 4.8.** Let $\mathcal{D}$ be an abelian group, and $\mathcal{E}$ a sub-semigroup of $\mathcal{D}$. Suppose that $\pi : \mathcal{E} \twoheadrightarrow \mathcal{P}$ is a surjective homomorphism, and that $\deg : \mathcal{P} \to \mathbf{Z}$ is a homomorphism. By abuse of notation, we will also use deg to denote the composition $\deg \circ \pi$ and its restriction to $\mathcal{E}$. We will refer to the image under deg of an element in $\mathcal{E}, \mathcal{D}$, or $\mathcal{P}$ as the degree of that element. We shall also assume that all nonzero elements of $\mathcal{E}$ have positive degree. The data of $\mathcal{E}, \mathcal{D}, \mathcal{P}, \pi$, and deg satisfying these conditions will be referred to collectively as a *divisor system*, and we will usually simply refer to the system as a whole as $\mathcal{D}$. Elements of $\mathcal{D}$ will be called *divisors*, elements of $\mathcal{E}$ will be called *effective divisors*, and elements of $\mathcal{P}$ will be called *divisor classes*. Two divisors $A, B \in \mathcal{D}$ will be called *equivalent* if $\pi(A) = \pi(B)$.

**Definition 4.9.** Suppose that $\mathcal{D}$ is a divisor system. Then for $A, B \in \mathcal{D}$, we shall write $A \geq B$ to denote $A - B \in \mathcal{E}$. For any $A \in \mathcal{D}$, $|A|$ will denote the set of all effective divisors equivalent to $A$. The *rank* of a divisor is $r(A) = \min\{\deg E : E \in \mathcal{E},\ |A - E| = \emptyset\} - 1$.

Note that the condition that $\deg(A) \neq 0$ for nonzero effective divisors is imposed to ensure that for divisors $A$ of degree 0, $r(A) = 0$ holds if and only if $\pi(A) = 0$.

Using this terminology, we can now formulate the necessary criterion. We point out that the Riemann-Roch criterion in [4] is phrased somewhat differently, but it is equivalent. The criterion simply expresses the fact that maximal divisors with empty linear series form a symmetric set in a single degree.

**Definition 4.10.** A divisor system is called *Baker-Norine* if there is a set $\mathcal{N}$ of divisors, all of the same degree, such that the following two conditions hold.

BN1 For any divisor $A$, $|A| = \emptyset$ if and only if $A \leq \nu$ for some $\nu \in \mathcal{N}$.

BN2 There exists a divisor $K$ such that $K - \mathcal{N} = \mathcal{N}$.

The set $\mathcal{N}$ is called the set of *moderators*, and the *genus* $g$ is the integer such that all moderators have degree $g-1$.

Observe that in a Baker-Norine divisor system, the genus and moderator set are uniquely determined (the moderator set must include *all* divisors of degree $g-1$). The divisor $K$ is not uniquely determined, but we will see momentarily that its divisor class is. (ALSO remark on tropical curves)

As we observed at the beginning of this section, the divisors on an algebraic curve give a Baker-Norine system. In fact, the Baker-Norine criterion is equivalent to the Riemann-Roch formula., due to the following theorem, originally proved by Baker and Norine [4].

**Theorem 4.11.** *A divisor system is Baker-Norine if and only if there exist an integer $g$ and a divisor $K$ that satisfy the Riemann-Roch formula for all divisors $A$:*

$$r(A) = r(K - A) + \deg(A) + 1 - g. \tag{2}$$

*Proof.* First, suppose that $g$, $K$ exist such that equation 2 holds. Then for any divisor $A$ of degree $g-1$, $r(A) = r(K-A)$, and property BN2 follows. Also, for any divisor $A$, $r(A) < 0$ if and only if $r(K-A) < g - 1 - \deg(A)$. By definition, this is true if and only if there is some effective divisor $E$ of degree $g - 1 - \deg(A)$ such that $r(K - A - E) < 0$. In this case, $\deg(K - A - E) = g - 1$. In other words, $r(A) = -1$ if and only if there exists some $\nu \in \mathcal{N}_{g-1}$ such that $\nu \leq K - A$, i.e. $A \leq K - \nu$. But since property BN2 holds, this is true if and only if $A \leq \nu$ for some $\nu \in \mathcal{N}_{g-1}$. Thus property BN1 holds as well, and the divisor system is Baker-Norine.

Conversely, suppose that the divisor system is Baker-Norine, for some $g, K$. Let $A$ be any divisor, and suppose that $E$ is some effective divisor such that $r(A - E) = -1$. The existence of $E$ can be used to construct a corresponding effective divisor $F$ for $(K - A)$, as follows. By property BN1, there exists some $\nu \in \mathcal{N}_{g-1}$ such that $A - E \leq \nu$, i.e. there is an effective divisor $F$ such that $A - E = \nu - F$. This is equivalent to $(K - A) - F = (K - \nu) - E$. Now, by property RR2, $K - \nu \in \mathcal{N}_{g-1}$, and thus by property RR1, $r((K - A) - F) = -1$. By definition, this implies that $r(K - A) \leq \deg(F) - 1$. Now, $\deg(F) = g - 1 + \deg(E) - \deg(A)$. If $E$ is chosen to have the smallest possible degree, namely $r(A) + 1$, then we obtain the following inequality:

$$r(K - A) \leq r(A) - \deg(A) - 1 + g. \tag{3}$$

Replacing $A$ with $K-A$ gives the reverse inequality, and establishes the Riemann-Roch formula. □

**Corollary 4.12.** *In a Baker-Norine divisor system, the integer $g$ is unique, and $K$ is unique up to equivalence.*

*Proof.* That $g$ is unique follows from the remark after definition 4.10. Suppose $K_1$, $K_2$ are two distinct divisors such that $\mathcal{N}_{g-1}$ is symmetric about both. Then formula 2 holds using either divisor, hence $r(K_1 - A) = r(K_2 - A)$ for all divisors $A$. In particular, $r(K_1 - K_2) = r(0) = 0$, and it follows that $K_1 - K_2$ is equivalent to 0. □

## 4.2 Tropical Riemann-Roch

The basic idea in [4], used to prove the Riemann-Roch theorem for graphs, is to identify a specific divisor in any given equivalence class that is extremal in a suitable sense, and demonstrate the condition BN1 for such extremal divisors. The same notion does not apply verbatim in the case of tropical curves, but can easily be adapted to it. We follow roughly the same approach as citeMZ. An alternative proof can be found in [12], where the Riemann-Roch theorem for tropical curves is deduced from the Riemann-Roch theorem for graphs.

For various reasons, it will be easier in the discussion that follows to exclude infinite edges.

**Lemma 4.13.** *Let $\Gamma$ be a tropical curve, and $\Gamma'$ a metric graph such that $\Gamma$ can be obtained by adding infinite edges to $\Gamma'$. Then Div$(\Gamma)$ is Baker-Norine if and only if Div$(\Gamma')$ is Baker-Norine.*

*Proof.* As observed in the paragraph before example 4.6, any divisor on $\Gamma$ is linearly equivalent to a divisor with no support on infinite edges, by simply moving all points in the divisor to the nearest vertex in $\Gamma'$. Hence we obtain a map $\mathrm{Div}(\Gamma) \twoheadrightarrow \mathrm{Div}(\Gamma')$ that induces an isomorphism of Picard groups and preserves effective divisors. Hence this map also preserves ranks of divisors. It follows that Riemann-Roch is true for $\Gamma$ if and only if it is true for $\Gamma'$, i.e. $\mathrm{Div}(\Gamma)$ is Baker-Norine if and only if $\mathrm{Div}(\Gamma')$ is Baker-Norine. □

Other proofs of the tropical Riemann-Roch theorem proceed by studying so-called $v$-reduced divisors, which are unique divisors in a given divisor class that are particularly easy to study. The analog for algebraic curves is as follows: given a divisor $A$ and a point $p$ on an algebraic curve, one can ask what the maximum order of vanishing (possibly negative) is for a meromorphic section of the line bundle $\mathcal{L}(A)$ that is holomorphic at all points except $v$. There is a unique section (up to scale) that achieves this maximum; its divisor is called *$v$-reduced*. Baker and Norine studied an analogous notion for graphs in [4], and used this notion to prove the Riemann-Roch theorem for graphs, and the same idea has been generalized to tropical curves. Given a tropical curve, a chosen point $v$, and a divisor class, there is a unique $v$-reduced divisor in this class. In this article, we define a weaker notion, which we call a divisor *tight at $v$*. Unlike $v$-reduced divisors, divisors tight at $v$ are not unique in their divisor class, however the notion is still strong enough to prove the Riemann-Roch theorem. We have adopted it here because we believe that it makes the proof slightly more comprehensible.

**Definition 4.14.** Let $\Gamma$ be a metric graph, and let $v \in \Gamma$, and suppose that $A, B$ are two divisors on $\Gamma$. Then $A$ is called *nearer to $v$* than $B$ if there exists a radius $r \geq 0$ such that all point within distance $r$ of $v$ in $\Gamma$ have at least as large a coefficient in $A$ as in $B$, and at least one point has a strictly larger coefficient in $A$. This relations is transitive. A divisor $A$ is called *tight at $v$* if it is effective away from $v$, and no other divisors in its equivalence class and effective away from $v$ is nearer to $v$ than $A$.

**Lemma 4.15.** *On a metric graph $\Gamma$, for each $v \in \Gamma$ and each divisor class, there exists a divisor in this class that is tight at $v$.*

*Proof.* First, we verify that given a divisor $A$, there is some $A'$ equivalent to $A$ that is effective away from $v$. It suffices to show that for any $p \neq v$, there is a rational function $f$ such that $(f) - p$ is effective away from $v$. Taking a finite sum of such functions, we will be able to find such a divisor $A'$, by successively canceling all negative coefficients (at the expense of more and more negative coefficients at $v$). Such a function $f$ can be constructed as a continuous piecewise linear function of the distance from $v$ in $\Gamma$ whose slope increases by a sufficient quantity at each distance where a vertex of $\Gamma$ occurs.

Now, given that such divisors $A'$ exist, restrict attention to those with the largest possible coefficient at $v$. Now, the degree of these divisors away from $v$ is fixed. By compactness, we may restrict to only those divisors whose point nearest to $v$ attains the minimum possible value, and also restrict to those divisors which attain this minimum at a particular chosen point. We may proceed in this fashion until each point in the divisor has been uniquely specified. The result must be a divisor equivalent to $A$ that is tight at $v$. □

**Definition 4.16.** An *ranking* $\rho$ of a tropical curve $\Gamma$ is a choice of orientation for all edges in some subdivision of $\Gamma$, with the property that there are no oriented cycles in $\Gamma$. The *rank divisor* of a ranking $\rho$ is the divisor $\nu_\rho$ on $\Gamma$, supported on the vertex set of the subdivision used to define $\rho$, whose coefficient at a point $p$ is one less than the number of outward-oriented edges incident to $p$.

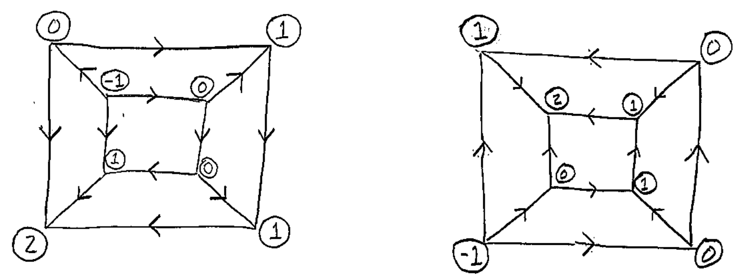


Figure 7: Rank divisors for two opposite rankings on a graph.

**Lemma 4.17.** *A rank divisor $\nu_\rho$ has degree $g-1$ and $|\nu_\rho| = \emptyset$. If $\rho'$ is the opposite ranking of $\rho$, then $\nu_\rho + \nu_{\rho'}$ is the canonical divisor.*

*Proof.* The degree of $\nu_\rho$ is simply $\sum_{v\in\Gamma}(\nu_\rho(v) - 1) = |E| - |V| = g - 1$, where $E, V$ are the edges and vertices of some subdivision of $\Gamma$ that includes all points in the support of $\nu_\rho$. Now, suppose $|\nu_\rho| \neq \emptyset$. Then there is some rational function $f$ such that $\nu_\rho + (f)$ is effective. The ranking $\rho$ generates a partial order on the vertices of and subdivision of $\Gamma$ that includes all places where the orientation changes direction. Among all vertices in a sufficient subdivision, consider the points that maximize $f$. Among these points, choose one point, $p$, that is minimal with regard to the order induced by $\rho$. Then the slope of $f$ is negative along each edge out of $p$ that is oriented away from $p$, and it is at most zero in all other directions. It follow that the coefficient of $p$ in $\nu_\rho + (f)$ is negative, and hence this divisor is not effective. It follows that $|\nu_\rho| = \emptyset$. The final statement follows immediately from definitions. □

The fact that $\nu_\rho + \nu_{\rho'}$ is the canonical divisor will be the final step in the proof of the Riemann-Roch theorem, and will explain why the canonical divisor functions as a dualizer.

**Lemma 4.18.** *For any $v$-reduced divisor $A$ that is not effective, there exists some rank divisor $\nu_\rho$ such that $A \leq \nu_\rho$.*

*Proof.* Let $G$ be a multigraph obtained by a subdivision of $\Gamma$ that includes all vertices in the support of $A$ as vertices. We shall construct an order on the vertices of $G$, which will given rise to a ranking $\rho$, by orienting each edge in the direction of the lower-ordered vertex. Let the highest-order vertex be $v$, and construct the rest of the order inductively, as follows. Suppose that at a given stage, the set of vertices already placed in the order is $S$, and let $T$ be the complement of $S$ in $V(G)$. Let $\ell$ be the length of the shortest edge between a vertex of $S$ and a vertex of $T$. Define a rational function $f$ that is equal to 0 on $S$, $\ell$ on $T$, and has slope 1 for the length $\ell$ segment beginning in $T$ for each edge between $S$ and $T$. Then $(f)$ is a divisor with positive coefficient for some point on each edge between $S$ and $T$, and whose coefficient at any vertex in $T$ is the number of edges from that vertex to a vertex in $S$. Then $A + (f)$ must have negative coefficient at some point in $T$, or else it would be a a divisor effective away from $v$ that is nearer to $v$ than $A$. This point must be a vertex $w$ in $T$; let $w$ be the next vertex in the order of $V(G)$.

Having constructed this order, and the associated orientation $\rho$ in this way, each vertex in $V(G)$ other than $v$ has more edges oriented towards it than its coefficient in the divisor $A$, by construction. But this means precisely that $A \leq \nu_\rho$ away from $v$. However, we assumed that $A$ was not effective, so the coefficient of $v$ in $A$ is at most $-1$, which is the coefficient of $v$ in $\nu_\rho$. Thus $A \leq \nu_\rho$, as desired. □

Combining these lemmas, we obtain the Riemann-Roch theorem for tropical curves.

**Theorem 4.19** (Tropical Riemann-Roch)**.** *Let $\Gamma$ be a tropical curve, with canonical divisor $K$ and genus $g$. Then for all divisors $A$,*

$$r(A) = r(K - A) + \deg(A) + 1 - g. \tag{4}$$

*Proof.* By lemma 4.13, it suffices to prove the result for metric graphs, so assume that $\Gamma$ has no infinite edges. Let $\mathcal{N}$ be the set of all divisors which are linearly equivalent to a rank divisor. By lemma 4.17, these are all divisors with empty linear series and in the same genus. Now, for any divisor $A$, if $A \leq \nu$ for some $\nu \in \mathcal{N}$, then clearly $A$ has empty linear series. Conversely, if $A$ has empty linear series, then upon fixing a vertex $v$, $A$ is linearly equivalent to a $v$-reduced divisor $A'$ by lemma 4.15, and by lemma 4.18, $A' \leq \nu$ for some $\nu \in \mathcal{N}$, and thus $A \leq A + (\nu - A')$, and $A + (\nu - A') \in \mathcal{N}$. Thus condition BN1 from definition 4.10 is satisfied for the set $\mathcal{N}$.

Now, observe that for any rank divisor $\nu_\rho$, if $\rho'$ is the ranking obtained by reversing all the orientations of $\rho$, it follows from the definition of $K$ that $\nu_\rho + \nu_{\rho'} = K$. It follows that $\mathcal{N}$ is symmetric about $K$, so condition BN2 from defintion 4.10 is satisfied, and hence Div$\Gamma$ is Baker-Norine. By theorem 4.11, the Riemann-Roch formula holds. □

We are not aware of an interpretation of the tropical Riemann-Roch theorem that views the quantity $r(A) - r(K - A)$ as an Euler characteristic, as it is often viewed in classical geometry. If such an interpretation could be found and generalized, then it could potentially be brought together with the general tropical intersection theory apparatus developed by Mikhalkin in [21] to prove a tropical Hirzebruch-Riemann-Roch theorem. This would be a marvelous development, but we are not currently aware of any progress in this direction.

# 5 Tropical plane curves

We now study tropical curves that arise as non-archimedean amoebas in $\mathbf{R}^2$. Such amoebas can be studied more simply not as the image of curves over $\mathbf{K}$ under valuation, but as the corner sets of tropical polynomials. Theorem 2.4 is the basic tool that permits this correspondence. Each tropical polynomial can be lifted to many polynomials over $\mathbf{K}$, all of which share properties with the tropical curve in $\mathbf{R}^2$. We begin with summarizing some notions from tropical algebra.

## 5.1 Tropical algebra and tropical projective space

**Definition 5.1.** The *tropical semifield* $\mathbf{T}$ is the set $\mathbf{R} \cup \{-\infty\}$ endowed with the following two operations.

- *Tropical addition*: $x \oplus y = \max(x, y)$.
- *Tropical multiplication*: $x \otimes y = x + y$.

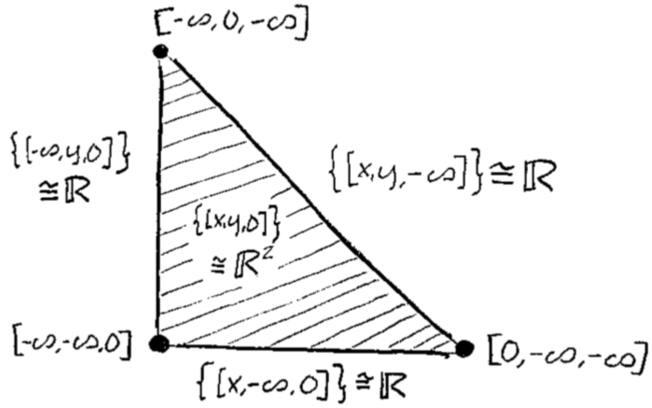


Figure 8: A stratification of the projective tropical plane, which is homeomorphic to a closed disc.

Observe that $\mathbf{T}$, with tropical addition and multiplication, satisfies all the axioms of a field (where the additive and multiplicative identities are $-\infty$ and 0), except for the existence of additive inverses. Such inverses clearly cannot exist because $x \oplus x = x$ for all $x \in \mathbf{T}$.

Some authors (for example, [25]) use a different convention, defining tropical addition to be given by maximum, rather than minimum. In this case, the underlying set includes $+\infty$ rather than $-\infty$ as the additive identity.

A *tropical polynomial* is simply an expression in some number of unknowns, constructed using tropical addition and multiplication. Tropical polynomials are piecewise-linear functions with integer slope where differentiable.

In this section, we shall only work in $\mathbf{R}^2$. However, there are various compactifications of the plane available; in particular, each compact toric surface gives a compactification. We shall not need these for our purposes, but we mention the most straightforward example to give a sense for where tropical plane curves might more naturally live.

**Definition 5.2.** The *tropical projective plane* $\mathbf{TP}^2$ is the set $\mathbf{T}^3 \backslash \{[-\infty, -\infty, -\infty]\}$ modulo tropical multiplication.

Observe that the tropical projective plane has the topology of a closed disc (see figure 8). In particular, the points at infinity form a topological boundary component. This contrasts with $\mathbf{CP}^2$, where the choice of the line at infinity is entirely arbitrary.

### 5.2 Tropical curves in $\mathbf{R}^2$

We now describe how tropical polynomials in two variables give rise to tropical curves in the sense of [11], along with a geometric realization in $\mathbf{R}^2$. More precisely, we will describe a way to associate, to each tropical polynomial in two variables, an abstract tropical curve $\Gamma$, together with a continuous map from the finite points of the curve to $\mathbf{R}^2$ such that piecewise linear functions on $\mathbf{R}^2$ pull back to rational functions on $\Gamma$.

For notational convenience, we shall denote by $\Gamma_{\text{fin}}$ the finite part of $\Gamma$.

Let $p(x, y) = \bigoplus_{(i,j)\in I}(c_{i,j} \otimes x^{\otimes i} y^{\otimes j})$ be a tropical polynomial in two variables. Here, $I$ is called the *monomial set*, and we assume that each $c_{i,j} \neq -\infty$. Of course, we could just as well take $I$ to

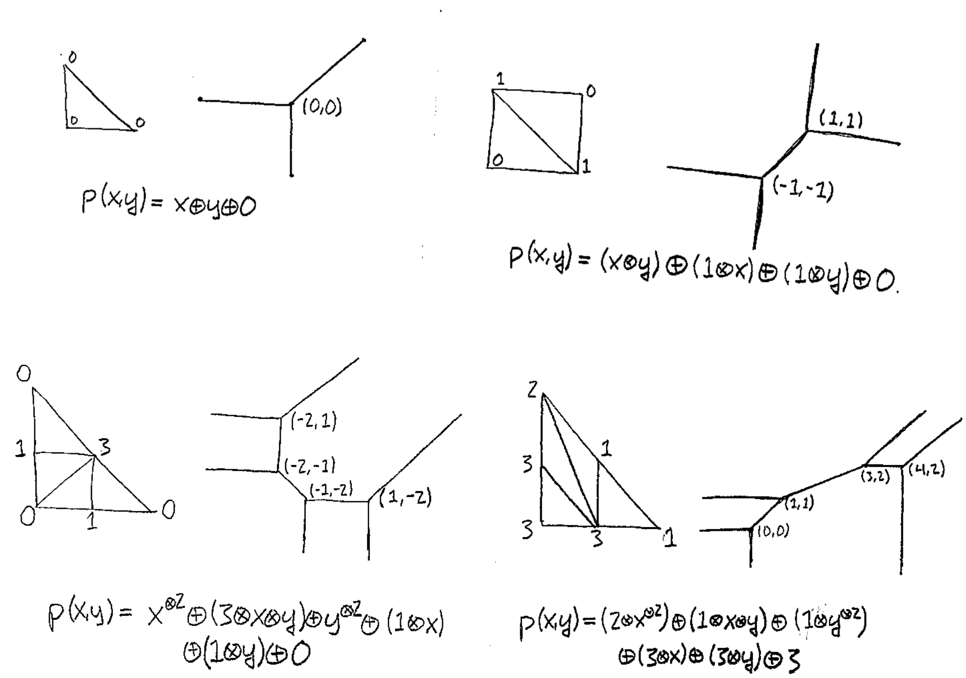


Figure 9: Four tropical curves, along with their Newton subdivisions and defining polynomials.

be $\mathbf{N} \times \mathbf{N}$ and stipulate that all but finitely many $c_{i,j}$ are $-\infty$. However, as we will see, the set $I$ of finite coefficients is very important to the behavior of the curve.

We define the tropical plane curve defined by $p$ (as a set) as follows.

**Definition 5.3.** Let $p(x, y)$ be as above. Then for all $(x, y) \in \mathbf{R}^2$, define the *valence* of $(x, y)$ with respect to be the number of monomials $c_{i,j} \otimes x^{\otimes i} y^{\otimes j}$ which achieve the value $p(x, y)$. The *tropical plane curve* corresponding to $p(x, y)$ is the closure in $\mathbf{R}^2$ of the set of points with valence greater than 1. A subspace of a tropical plane curve is called an *edge* if it is a maximal subspace homeomorphic to an open interval, and points that do not lie on edges are called *vertices.*

Figure 9 shows the tropical curves in $\mathbf{R}^2$ corresponding to four different tropical polynomials. As we shall do for the rest of this paper, we have drawn the Newton subdivision to the left of each curve. The Newton subdivision is a convenient way to represent the coefficients of a tropical polynomial, as we shall see.

**Definition 5.4.** Let $p(x, y)$ be a tropical polynomial, and suppose that the indices $(i, j)$ of the monomial set are plotted in $\mathbf{Z}^2$. Then the convex hull of these points is called the *Newton polygon* of $p$. Suppose in addition that an edge is draw between all pairs of indices such that the corresponding monomials simultaneously achieve the value $p(x, y)$ somewhere in the plane. The result is a subdivision of the Newton polygon into small polygons. This subdivision is called the *Newton subdivision*, and the smaller polygons are called the *faces* of the subdivision.

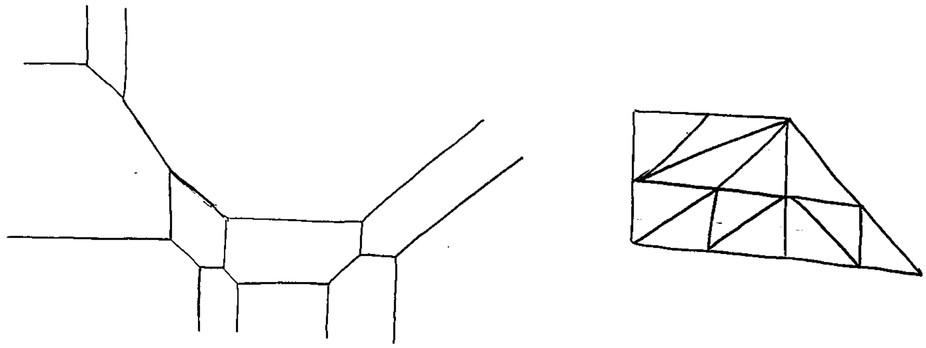

Figure 10: A plane tropical curve of genus 2.

When drawing the Netwon polygon, we typically label vertices with the coefficient $c_{i,j}$, as in figure 9. The resulting picture is a useful visualization tool, because it gives a dual graph for the tropical plane curve, in the sense that the faces of the subdivision correspond to vertices of the curve, and the edges correspond to edges of the curve. The edges on the boundary of the Newton polygon correspond to infinite edges. We shall see that several other attributes, including the genus of the curve and various notions of multiplicity, are most easily expressed in terms of the Newton subdivision.

Figure 10 shows a more complicated tropical curve, of genus 2. We point out that the general shape of the curve can be completely read off of the Newton subdivision; in this case, we have not even bothered to choose the coefficients. This curve is also a good illustration of the genus formula, which we shall see in section 5.3.

The duality between the Newton polygon (labelled with coefficients) and the plane curve can be observed as follows. Given a point $(x, y) \in \mathbf{R}^2$, we obtain a linear form $(i, j, c_{i,j}) \mapsto c_{i,j} + ix + jy$ on $\mathbf{Z}^2$ which described how large each possible monomial is. This form defines a plane $\Lambda$ through the origin in $\mathbf{R}^3$. Now, if we imagine putting the Newton subdivision into $\mathbf{R}^3$ by raising each vertex $(i, j)$ to heigh $c_{i,j}$, the largest monomial, at the point $(x, y)$, will correspond to the vertex that rises the highest above this plane. Hence the edges in the plane curve will correspond precisely to pencils of planes that are equal on some set of coefficients; these coefficients must, then, all lie on a line in $\mathbf{R}^3$. The vertices of the plane curve correspond to a plane that contains three or more vertices that do not lie on a line; this will precisely be a face of the Newton subdivision. Finally, the slope of any edge in the plane curve will be perpendicular to the slope of the corresponding edge in the Newton subdivision; this is a straightforward calculation. In many ways, the Newton subdivision appears to be the most efficient way to quickly visualize and draw tropical plane curves. In addition, in provides the correct combinatorial framework for some enumerative problems in tropical geometry, as in [19].

All of the examples of tropical plane curves that we have presented so far have had had fully subdivided Newton polygons (i.e. the subdivision breaks the polygon into triangles of area $\frac{1}{2}$). However, consider what happens when this is not the case. For example, we can deform the bottom two tropical curves in figure 9 by changing a single coefficient to bring two of the infinite tentacles into coincidence. The result is shown in figure 11. Clearly we should regard the edges that are drawn double in this figure as being edges of multiplicity 2. Thus we make the following

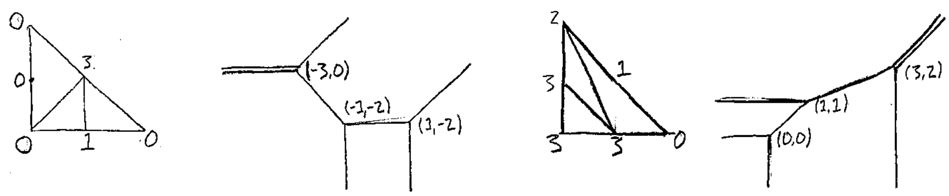


Figure 11: The tropical quadrics from figure 9, upon having two tentacles brought together to produce an edge of higher multiplicity.

definition.

**Definition 5.5.** The *multiplicity* of an edge in a hypersurface is the val$(p) - 1$, where $p$ is any point on the interior of the edge.

Edge multiplicity is very easy to read off of the Newton polygon: each edge in the hypersurface passes through some number of vertices of the polygon, and the multiplicity is one less than this number.

An informal justification for this definition of multiplicity can be seen by considering how an edge with multiplicity could arise from a degeneration of an amoeba. Suppose that an edge arises due to the equality of the terms corresponding to monomials $x^iy^j, x^{i+d}y^{j+e}, \ldots x^{i+md}y^{j+me}$, where $m$ is the multiplicity. Then $d$ and $e$ are relatively prime, or else there would be more monomials in between in the Newton polygon. Thus the edge in question arises from the degeration of a part of the curve that eventually comes to look like it is given by an equation $0 = x^iy^j \cdot q(x^dy^e)$, where $q$ is a polynomial of degree $m$ in one variable. But this equation factors, and once points that do not lie in $(\mathbf{C}^*)^2$ are thrown out, what remains are $m$ irreducible curves (one for each root of $q$). Hence in the limit, the resulting edge in $\mathbf{R}^2$ corresponds to $m$ different curves layering on top of each other. All this can be understood more rigorously be using Puiseux series, and the critical technical ingredient is Hensel's lemma; we will discuss how this works in section 5.5.

Thus we see that in order to specify a tropical plane curve, we should really give not only the set in $\mathbf{R}^2$, but also the edge multiplicities. The reader can verify that these data unique determine the polynomial defining the curve, up to tropical multiplication by a scalar. We can now describe how to associate to a tropical plane curve a map from an abstract tropical curve.

**Definition 5.6.** Given a tropical polynomial $p(x, y)$ that defines a tropical plane curve $T$, we define a tropical curve $\Gamma$ as follows. For each vertex in$T$, $\Gamma$ has a finite vertex. For each edge of multiplicity $m$ between finite points in $T$, $\Gamma$ has $m$ edges between the corresponding finite points. For each infinite edge of multiplicity $m$ in $T$, $\Gamma$ has $m$ infinite edges originating from the corresponding finite vertex. If $T$ has an edge with no finite end, first subdivide this edge with a finite vertex and proceed as already described.

For each edge $[0, \ell]$ in $\Gamma$, we define a linear map this edge to $T$ such that $f(0)$ is the finite vertex corresponding to the vertex 0, and the slope of $f$ is precisely the primitive integer vector pointing in the direction of the edge in $T$. Taken together, this defined a continuous rational map $f : \Gamma_{\text{fin}} \to T$.

This definition seems rather complicated, but it actually quite simple: the plane curve essentially gives a model for the metric graph. The only wrinkles are that edges with multiplicity must be

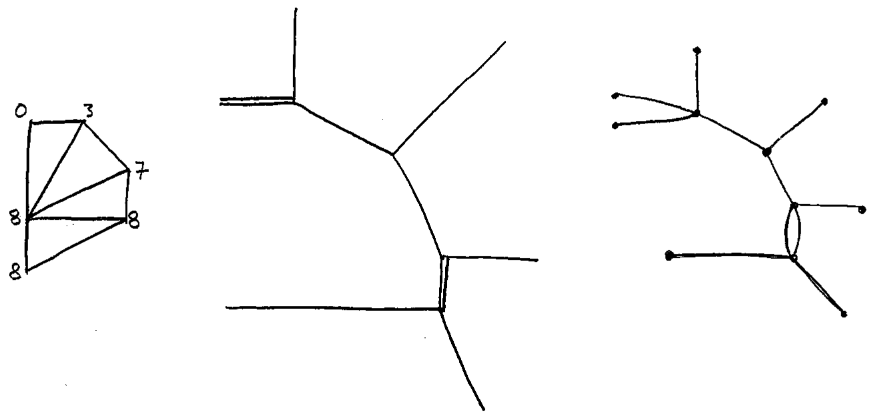


Figure 12: A tropical curve with multiple edges and its corresponding abstract tropical curve.

split into several edges between the same vertices, points at infinity must be split to avoid having several infinite edges go to the same point at infinity, and edge lengths must be assigned in such a way that the map from $\Gamma$ to the plane curve is piecewise linear with integer slope.

Figure 12 depicts a tropical plane curve with its Newton subdivision, and the corresponding abstract tropical curve. Notice that the plane curve has two double edges; the one going to infinity becomes two edges to two different infinite points, while the finite one becomes two edges between the same two vertices. Although this may seem like a curious convention (as opposed to allowing the doubled infinite edge to become two edges to the same infinite vertex), it is actually very convenient. For example, we shall see that the genus formula discussed in the following section would not hold in nearly as nice a form if we used a different convention.

To demonstrate that our convention on lengths of edges and edge multiplicity makes sense, we prove the following proposition (which we should certainly hope to be true if the definition properly mirrors the degenerations of amoebas we are trying to formalize). In [11] and elsewhere, this proposition is called the *balancing condition*, and is phrased in different language that does not make use of abstract tropical curves. A second, perhaps more convincing, vindication of this convention will come in the following section.

**Proposition 5.7.** *Suppose $f : \Gamma_{fin} \to T$ is as above. The functions $x$ and $y$ on $\mathbf{R}^2$ pull back to $\Gamma$ as rational functions whose divisors are supported at infinite points only.*

*Proof.* It is clear that $x$ pulls back to each edge of $\Gamma$ as a linear function, so its divisor must be supported on the vertex set. Consider one vertex, $v$. For a sufficiently small neighborhood of $v$ in $\mathbf{R}^2$, $T$ consists of several rays emanating from $v$. In between each two rays is a region in which a unique monomial of $p$ achieves $p(x, y)$. Suppose that two adjacent such regions correspond to monomials $c_{i_1,j_1} \otimes x^{i_1} \otimes y^{j_1}$ and $c_{i_2,j_2} \otimes x^{i_2} \otimes y^{j_2}$. Then the edge dividing these two regions has slope $\frac{1}{m}(j_2 - j_1, i_1 - i_2)$, where $m$ is the multiplicity of the edge. Thus the outward slope of the pullback of the function $x$ is $\frac{1}{m}(j_2 - j_1)$. Now, summing for all such edges in $\Gamma$ results in summing $j_2 - j_1$ for each edge in $T$ leaving $v$, which clearly cancels. Thus the divisor of $x$ must be supported only at infinity. The same reasoning will apply to the pullback of the function $y$ to $\Gamma$. $\square$

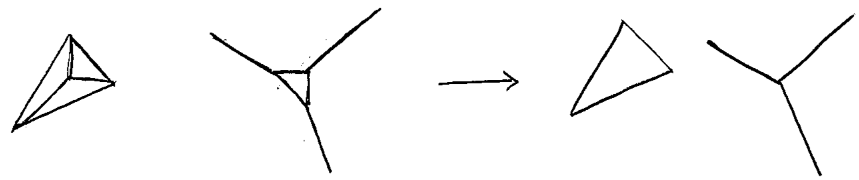

Figure 13: Degeneration to the singular point of the curve in figure 12, together with the corresponding part of the Newton subdivision.

## 5.3 Calculation of the genus

One would hope that the tropical curve obtained by degenerating an amoeba would have the same genus as the original curve. This following theorem shows that this is nearly true, and gives a criterion to see where it fails.

**Theorem 5.8.** *Suppose that $p(x, y)$ is a tropical polynomial, associated to a map $f : \Gamma \to T \subset \boldsymbol{R}^2$ from an abstract tropical curve. Then the genus of $\Gamma$ is equal to the number of vertices on the interior of the Newton polygon of $p$ that are not on the interior of any face of the Newton subdivision.*

We shall prove this theorem momentarily, but we first make some remarks. Referring back to figures 10 and 12 give two examples of this theorem. The curve in figure 10 uses all of the vertices in its Newton polygon for subdivision. The curve in figure 12, on the other hand, has one vertex whose corresponding face in the Newton subdivision has an interior point (the point is not marked, but lies in the triangle bounded by coefficients $3, 7, 8$). We may regard the corresponding point in the curve as a singular point, which causes the genus of the tropical curve to drop. Figure 13 illustrates how a cluster of nonsingular points (which would have contributed 1 to the genus of the plane curve) can degenerate to the singular point of the curve in figure 12.

Just as in the case of algebraic curves, to each such "singularity" we may associate an integer (in this case, the number of vertices inside the corresponding face of the Newton subdivision) which measures, precisely, the contribution of the singularity to the genus of the tropical curve. This suggests the following definition, to carry on the analogy to algebraic curves. We are not aware whether the following definition is standard in the literature.

**Definition 5.9.** The *arithmetic genus* of a tropical plane curve is the number of points in the interior of the Newton polygon of the defining tropical polynomial. The *geometric genus* of the curve is the genus of the abstract tropical curve $\Gamma$.

In [19], tropical plane curves whose Newton subdivisions completely subdivide the Newton polygon into triangles of area $\frac{1}{2}$ are called *smooth*. So we see that for a smooth curve, the genus is precisely the number of interior points of the Newton polygon. Hence we have the following two calculations, which agree with what we expect from algebraic curves of degree $d$ in the plane and of bidegree $(d, e)$ on $\mathbf{P}^1 \times \mathbf{P}^1$, respectively.

**Corollary 5.10.** *A smooth tropical plane curve which has monomials of all degrees up to $d$ has genus $\binom{d-1}{2}$.*

**Corollary 5.11.** *A smooth tropical plane curve which has monomials of all bidegrees that are componentwise less than or equal to* $(d, e)$ *is* $(d-1)(e-1)$*.*

We now give the proof of the theorem, which is essentially an exercise in Pick's theorem.

*Proof of theorem 5.8.* Let the faces of the Newton subdivision be $F_1, \ldots, F_n$. Let $I_i, B_i$ be the number of verticies on the interior and boundary of the face $F_i$, respectively. Let $v_i$ be the finite vertices of $\Gamma$ corresponding to face $F_i$. Let $w_1, \ldots, w_m$ be the infinite vertices of $\Gamma$. Then the number of vertices of $\Gamma$ is $n+m$ and the number of edges is $\sum_{v \in V(\Gamma)} \frac{1}{2}\mathrm{val}(v) = \sum_{i=1}^{n} \frac{1}{2}\mathrm{val}(v_i) + \frac{1}{2}m$, so the genus of $\Gamma$ is $\sum_{i=1}^{n} \frac{1}{2}\mathrm{val}(v_i) - \frac{1}{2}m - n + 1 = \sum_{i=1}^{n}(\frac{1}{2}\mathrm{val}(v_i) - 1) - \frac{1}{2}m + 1$. Now, $\mathrm{val}(v_i)$ is the number of edges around the face $F_i$ counted with the correct multiplicity, which is simply $B_i$. Let $A_i$ be the area of the face $F_i$. By Pick's theorem, $A_i = I_i + \frac{1}{2}B_i - 1$. Now, using this, it follows that the genus of $\Gamma$ is equal to $\sum_{i=1}^{n}(A_i - I_i) - \frac{1}{2}m + 1 = A - \sum_{i=1}^{n}(I_i) - \frac{1}{2}m + 1$, where $A$ is the area of the Newton polygon. Now, observe that applying Pick's theorem to all of the Newton polygon, and using the fact that $m$, the number of infinite vertices, is equal to the number of vertices on the boundary of the polygon, it follows that $A - \frac{1}{2}m + 1$ is the number of vertices on the interior of the Newton polygon. It follows that the genus of $\Gamma$ is the number of of vertices inside the Newton polygon that are not inside any of the faces of the subdivision. □

## 5.4 Stable intersection and the tropical Bézout theorem

One of the very convenient aspects of tropical plane geometry is that the intersection of two curves, as a set of points with multiplicity is always well-defined, even when intersecting a curve with itself. To distinguish this notion from the set-theoretic intersection of two tropical curves, we define the notion of stable intersection, as follows.

**Definition 5.12.** Let $T_1, T_2$ be two tropical plane curves, associated to maps $f_i : (\Gamma_i)_{\mathrm{fin}} \to \mathbf{R}^2$ from abstract tropical curves to the plane. Then the function $p_2$ pulls back to a rational function on $\Gamma_1$, with a divisor $A \in \mathrm{Div}(\Gamma_1)$. Then the *stable intersection* of the $\Gamma_1, \Gamma_2$ in $\mathbf{R}^2$ is the divisor $A$, restricted to the finite part of $\Gamma_1$ and pushed forward to $\mathbf{R}^2$.

From this definition, it is clear that the stable intersection is a finite number of points (counted with some multiplicities). However, the definition has the defect that it is not immediately clear that it is symmetric (i.e. that the same multiset is obtained if $T_1, T_2$ are interchanged). For now, we shall regard stable intersection as being a notion defined only for ordered pairs of tropical polynomials. After we have proven the Bézout theorem, however, we will prove in corollary 5.16 that in fact the stable intersection does not depend on the order that the two polynomials are given. After that, we may regard it as a symmetric notion.

Stable intersection can also be defined using perturbations: if the images of the two curves in $\mathbf{R}^2$ do not intersect transversely, then they can be perturbed slightly to produce a transverse intersection. Then, taking a limit as this perturbation returns to the original polynomials, one obtains the stable intersection. We have chosen to instead use the notion of rational functions on tropical curves, because we believe that it reduces the technicalities involved in proving that the notion is well-defined.

Figure 14 shows the stable intersection of two tropical lines whose ordinary intersection is not discrete. Figure 15 shows a single tropical curve intersecting a line in several different ways. Multiplicities are indicated next to intersections.

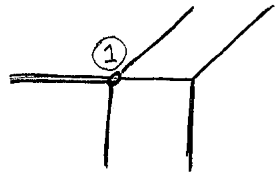


Figure 14: The stable intersection of two tropical lines.

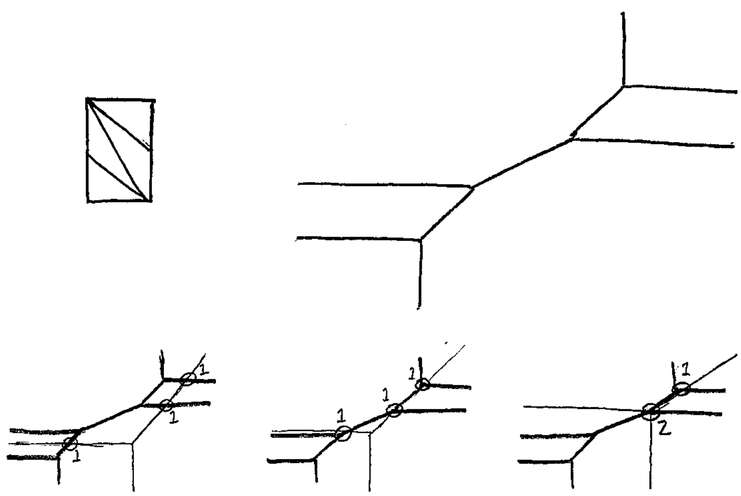


Figure 15: A tropical curve and its stable intersection with several different tropical lines.

Stable intersection can be visualized geometrically by observing that tropical curves are given by the "creases" in the graphs of piecewise-linear functions. Hence when intersecting two curves, we should only include the points on one curve which can "see" the crease of the function defining the other curve. In the case of the two lines intersecting in figure 14, the two functions crease in exactly the same direction, so the crease on the large set-theoretic intersection cannot be seen from the lines themselves, except at the point of stable intersection.

*Observation* 5.13. The same idea that underlies stable intersection can also be used very fruitfully for interpolation. The simplest example of interpolation is determining a tropical line through two chosen points. Such a line always exists, but is not always unique. However, a method is described in [25] which defines a unique line through two points in a natural way. The approach is closely related to the definition of stable intersection, which is also discussed in the same paper. Similarly, any five points define a unique quadric, in a suitable sense. An implementation of such tropical interpolation techniques and a discussion of some applications can also be found in [25].

Upon defining stable intersection, we are able to prove a tropical version of the Bézout theorem:

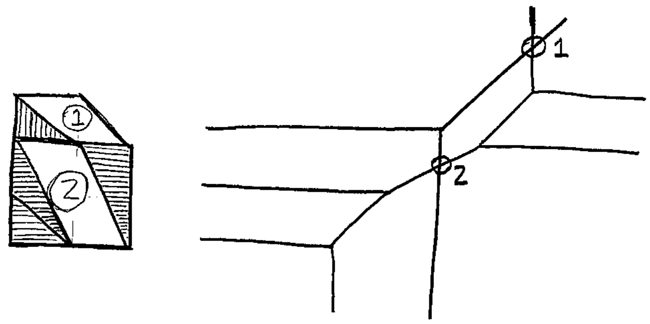


Figure 16: The computation of mixed volume for the intersection of a line and bidegree $(2,1)$ curve.

the degree of the stable intersection of two tropical curves depends only on the coefficient sets of the two polynomials.

**Theorem 5.14** (Tropical Bézout)**.** *If $p_1, p_2$ are tropical polynomials in two variables with coefficient sets having Newton polygons $N_1, N_2$, then the degree of the stable intersection of the corresponding tropical curves $T_1, T-2$ is equal to the mixed volume of $N_1, N_2$. That is, the number of intersections is $Area(N_1+N_2) - Area(N_1) - Area(N_2)$. Here we intend $N_1 + N_2$ to mean $\{x+y : x \in N_1, y \in N_2\}$.*

Before proving the theorem, we illustrate it with figure 16, which shows the computation of the relevant mixed volume for the intersection of a line and a bidegree $(2,1)$ curve shown in figure 15. The Newton subdivision shown is for the union of the two curves, and includes all the faces of the line (vertical hatching) and the other curve (horizontal hatching). What is left consists of a total area of 3. Each of the two remaining faces has area equal to the intersection multiplicity at that point.

*Proof.* Suppose that $p_1, p_2$ are tropical polynomials with Newton polygons $N_1, N_2$, and the rational function $p_2|_{\Gamma_1}$ on the tropical curve $\Gamma_1$ of $p_1$ is given by pulling back the function $p_2$ to $\Gamma_1$.

First, we prove that the number of intersections (with multiplicities) does not depend on the particular polynomial $p_2$, once $N_2$ has been fixed. To see this, observe that for a given Newton polygon for a polynomial $p$, there exists some radius $R$ such that for $(x,y)$ outside a circle of radius $R$, the slope of the function $p(x,y)$ does not depend on the choice of the coefficients of $p$. Then sufficiently far along the infinite edges of $\Gamma_1$, the slope of the function $p_2|_{\Gamma_1}$ is determined by the polygon $N_2$. Thus the portion of the divisor $(p_2|_{\Gamma_1})$ supported on the infinite part of $\Gamma_1$ is independent of the choice of $p_2$. Since the part of the divisor that is supported on the finite part of $\Gamma_1$ is linearly equivalent to the additive inverse of the part supported on the infinite part, the divisor class of the finite part of the divisor is independent of $p_2$. In particular, the degree of the finite part is fixed. It follows from this that the number of points in the stable intersection in $\mathbf{R}^2$, counted with multiplicities, is independent of the choice of $p_2$.

Thus in order to calculate this number, we may begin by assuming that $p_2$ is chosen as a general polynomial with polygon $N_2$. In particular, we may assume that all points of intersection of the two curves in $\mathbf{R}^2$ consist of transverse intersections of two line segments, meeting at a point in the relative interior of both line segments. Now, consider the tropical curve given by $p_1 \otimes p_2$. The Newton polygon of this polynomial will be $N_1 + N_2$. The faces of the Newton subdivision will all correspond to either vertices of $\Gamma_1$, vertices of $\Gamma_2$, or points of intersection of the two. The vertices of $\Gamma_i$ will have faces identical to the corresponding faces in the subdivisions of $N_i$, thus they will account for a portion of the area of $N_1 + N_2$ equal to the sum of the areas of $N_1$ and $N_2$. On the other hand, the intersection of two lines with slopes (multiplied by multiplicities) $(x_1, y_1), (x_2, y_2)$ will account for a face in the subdivision of $N_1 + N_2$ with area $\det\left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right)$. As is easy to verify, this is precisely equal to the intersection multiplicity of these two edges. Hence the total number of intersections, counted with multiplicity, is the difference between the total area of $N_1 + N_2$ and the sum of the areas of $N_1$ and $N_2$, as claimed. □

**Corollary 5.15.** *The number of intersections of the two curves, counted with multiplicity, is symmetric between the two polynomials.*

**Corollary 5.16.** *The stable intersection, as a multiset, is symmetric between the two polynomials.*

*Proof.* Let $p$ be any point in the tropical plane. Then the multiplicity of the intersection of the two curves at $p$ (possibly 0) can be calculated by restricting attention to the polynomials $p_1, p_2$ in a small neighborhood of $p$. In this neighborhood, we may assume that both curves are simply stars (several rays emanating from $p$). The stable intersection of these two stars must be supported only at $p$. By the previous corollary, the multiplicity of $p$ in this intersection does not depend on the order the polynomials are presented. Hence the multiplicity of $p$ in the original intersection also does not depend on the order that they were presented. □

*Observation* 5.17. As we see in the proof of theorem 5.14, the number of intersections of two curves can be calculated as an inclusion-exclusion calculation. This bears resemblance to a sheaf-theoretic proof of the classical Bézout theorem (see [5]), which we have deliberately attempted to mimic here, which proves that the number of intersections of two curves $C, D$ on a surface $S$ is given by

$$\#(C \cap D) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S(-C - D)). \tag{5}$$

It would be interesting to see whether equation 5 can be viewed as a sort of inclusion-exclusion as well, and indeed whether there is a tighter analogy between these two arguments that might even be able to be made precise.

We now consider some examples to illustrate the tropical Bézout theorem for some familiar Newton polygons.

*Example* 5.18. Suppose that $p(x, y)$ is independent of $y$. This describes a tropical plane curve that is the disjoint union of several vertices lines. The Newton polygon is just a line of area 0. The intersection of any two such curves is 0 (indeed, they can be pulled away from each other). On the other hand, the intersection of two curves, one a set of horizontal lines, and one a set of vertical lines, is simply $de$ ($d$ and $e$ being the respective numbers of lines), as can be seen from computing the area of the sum of these two segments. Of course none of this should be surprising.

*Example* 5.19. If $p_1, p_2$ are general polynomials of degrees $d_1$ and $d_2$, then their Newton polygons are simple right triangles as shown in figure 5.19. The mixed volume we obtain in this case is $\frac{1}{2}(d_1 + d_2)^2 - \frac{1}{2}d_1^2 - \frac{1}{2}d_2^2 = d_1 d_2$. This is of course the result we would expect, from the classical Bézout theorem. This is illustrated in figure 17.

*Example* 5.20. Suppose $p_1, p_2$ are general polynomials of bidegrees $(d_1, e_1)$ and $(d_2, e_2)$. Then their Newton polygons, as well as the sum of their Newton polygons, are rectangles. The mixed volume is $(d_1 + d_2)(e_1 + e_2) - d_1 e_1 - d_2 e_2 = d_1 e_2 + d_2 e_2$. This is the result that we would expect for intersections of algebraic curves on a $\mathbf{P}^1 \times \mathbf{P}^1$. This is illustrated in figure 18.

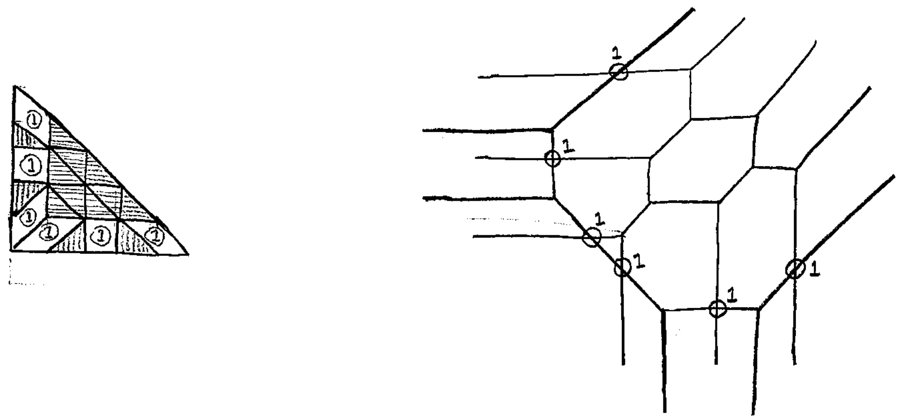

Figure 17: The intersection of a conic and a cubic in the plane, and the calculation of the relevant mixed volume.

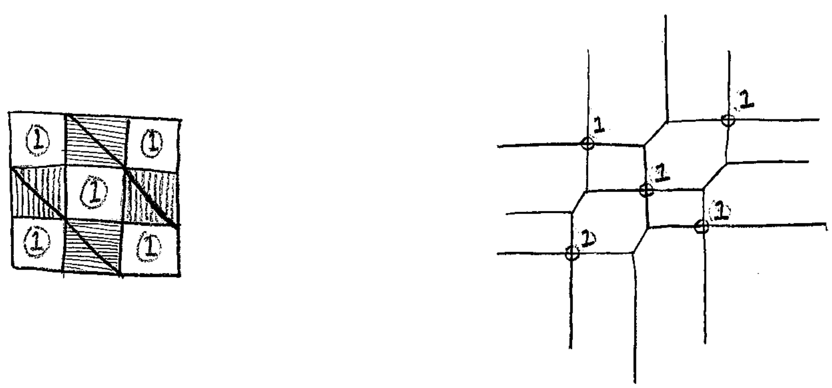

Figure 18: The intersection of a tropical bidegree $(2, 1)$ curve and a tropical bidgree $(1, 2)$ curve.

*Example* 5.21. As a final example, revisit figure 10, which has a comparatively nonstandard Newton polygon. It intersects a tropical line in four points. It may be amusing to compute its intersection numbers with various other types of tropical plane curves.

Before moving on, we briefly observe that the shape of the Newton polygon of a tropical plane curve is sometimes referred to as its *toric degree*. Just as intersections of algebraic curves in the projective plane are entirely controlled by degree, intersections of tropical plane curves are controlled by toric degree. As we see from the proof of the tropical Bézout theorem, knowing the toric degree is equivalent to knowing the portion of the divisor of $x^{\otimes i}y^{\otimes j}$ that is supported at the infinite points of the curve. As the name suggests, the toric degree of a curve described which toric surface the (algebraic) curve most naturally inhabits. For example, curves with Newton polygons that are right triangles naturally inhabit the projective plane, while curves with square Newton polygons naturally inhabit the quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$. We shall not discuss any specifics about toric surfaces in this article; a detailed account of this perspective can be found in [16].

## 5.5 Classical Bézout from tropical Bézout

One may well demand to know whether results such as the tropical Bézout theorem, or other tropical analogs of classical questions, can bear on classical algebraic geometry, or if they must be studied only for whatever independent interest they might have. In fact, there have been several applications (one is discussed in the next section) of tropical plane curves to classical geometry; in all such cases it is necessary to prove a suitable *correspondence theorem*. We illustrate one such correspondence theorem by describing how the classical Bézout theorem can be deduced from the tropical Bézout theorem (and more generally, the same correspondence can be used to calculate intersections on any toric surface).

The correspondence theorem that we shall discuss is only a special case of the general theorem that would be needed to prove Bézout: the case where the two tropical curves only meet at points on the interior of their edges. Of course this will suffice to prove the classical result for general curves, but the notion of stable intersection can be used to prove a more robust statement if desired. We shall not give a complete proof of this theorem, but we attempt at least to indicate the main points and the places where a complete proof becomes more technical. We have included this incomplete argument mainly because we believe that it provides excellent intuition for our notions of edge multiplicity and intersection multiplicity, by showing, at least in sufficiently general situations, what they actually correspond to in terms of the original curves.

**Theorem 5.22.** *Let $C_1, C_2$ be two plane curves over $\boldsymbol{K}$, and let $T_1, T_2$ be their non-archimedean amoebas in $\boldsymbol{R}^2$. Suppose that $T_1, T_2$ only intersect at points on the interiors of their edges. Then each intersection of $T_1, T_2$ corresponds to the same number of intersections of $C_1, C_2$ as the intersection multiplicity defined in the previous section.*

The idea behind this theorem is that the solutions in $\mathbf{R}^2$ can be bootstrapped back to solutions over the Puiseux field by specifying them more and more and more closely (in the topology given by the valuation). The necessary technical ingredient is the following generalization of Hensel's lemma.

**Lemma 5.23.** *Let $p(x, y), q(x, y)$ be two polynomials over a complete discrete valuation ring $R$, and let $\bar{p}(x, y), \bar{q}(x, y)$ be their reductions modulo the maximal ideal (as polynomials over the residue field $k$). Suppose that $(\bar{x}, \bar{y}) \in k^2$ is a solution for $\bar{p}$ and $\bar{q}$, and that the matrix of derivatives $\begin{pmatrix} \bar{p_x} & \bar{p_y} \\ \bar{q_x} & \bar{q_y} \end{pmatrix}$ is* *nonsingular at the point $(\bar{x}, \bar{y})$. Then there is a unique solution $(x, y) \in R^2$ to the polynomials $p, q$ reducing to $(\bar{x}, \bar{y})$.*

*Proof.* This is (a special case of) exercise 7.26 in [10]. If David Eisenbud is allowed to leave it as an exercise to the reader, then so am I. $\square$

For our purposes, $R$ is the ring $\mathbf{C}[[t]]$, and $k$ is $\mathbf{C}$. Now, given an intersection point $(u, v)$ of the two tropical curves, we can assume without loss of generality that $(u, v) = (0, 0)$ by first replacing $t$ by $t^k$ for $k$ sufficiently large that $u, v$ become integers, and then translating to the origin. Let us also choose $k$ sufficiently large that $p, q$ have no fractional exponents in their coefficients. By rescaling $p$ and $q$ by some power of $t$, we may also assume that the valuation that is achieved at $(0, 0)$ is 0. Now, reducing $p$ and $q$ modulo $t$ will remove all monomials that are not tying for the maximum valuation at the origin. The remaining terms have exponents in an arithmetic progression, as follows.

$$\begin{aligned}\bar{p}(x, y) &= c_0 x^i y^j + c_1 x^{i+a} y^{j+b} + \cdots + c_m x^{i+ma} y^{j+mb} \\ \bar{q}(x, y) &= d_0 x^k y^l + d_1 x^{k+c} y^{l+d} + \cdots + d_n x^{k+nc} y^{l+nd}.\end{aligned}$$

Here $(a, b)$ and $(c, d)$ are primitive integer vectors (i.e. the arguments of each have no common factor) corresponding to the slope of the corresponding edge in the Newton polygon, while $m, n$ are the precisely the multiplicities of the two edges that are meeting at the origin. We require that $c_0, c_m, d_0, d_n$ are all nonzero, but the other coefficients may be zero.

Now, we are observe that this means that $\bar{p}, \bar{q}$ can be written in the following form, for $r, s$ degree $m, n$ polynomials of one variable, respectively.

$$\begin{aligned}\bar{p}(x, y) &= x^i y^j r(x^a y^b) \\ \bar{q}(x, y) &= x^k y^l s(x^c y^d).\end{aligned}$$

We are only interested in solutions where both coordinates are nonzero, since otherwise the valuation would not be 0. Hence the number of solutions to this pair of equations in $(\mathbf{C}^*)^2$ is the number of solutions to $r(x^a y^b) = s(x^c y^d) = 0$.

Forgive us now for making an assumption: we shall assume that $r$ and $s$ have no multiple roots. If this is the case then a straightforward calculation will show that the derivative matrix of $\bar{p}, \bar{q}$, evaluated at any solution $(x, y)$, will be nonzero so long as $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| \neq 0$, which is simply means that the two tropical edges in question intersect transversely. Hence each solution to $r(x^a y^b) = s(x^c y^d) = 0$ will give exactly one point in the intersection of $C_1$ and $C_2$ over $\mathbf{K}$. But $r$ has precisely $m$ roots, $s$ has precisely $n$ roots, and for each of the $mn$ choices $(\mu, \nu)$ for $(x^a y^b, x^c y^d)$, choosing the actual values $(x, y)$ amounts to solving the equation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \begin{pmatrix} \log(x) \\ \log(y) \end{pmatrix} = \begin{pmatrix} \log(\mu) \\ \log(\nu) \end{pmatrix}$. Of course, all the logarithms are only well-defined modulo $2\pi i$. The lattice of possible branches for the right hand side has its area shrunk by a factor of the determinant $(ad - bc)$ under the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}$, hence the number of possible solutions modulo $2\pi i$ is precisely $(ad - bc)$.

Summing up, we see that the number of intersections of the curves $C_1, C_2$ that are send by the valuation map to this particular point in the amoeba is precisely $mn(ad - bc)$. Indeed, this is precisely the suggested intersection multiplicity that is used in the tropical Bézout theorem.

## 5.6 Enumerative geometry of tropical plane curves

We shall briefly describe in the section that work of Mikhalkin [19], which is an excellent application of plane tropical curves to classical algebraic geometry. Recall the following invariants.

**Definition 5.24.** Let $g, d$ be nonnegative integers. The the *Gromov-Witten invariant* $N_{g,d}$ of $\mathbf{P}^2$ is the number of curves of geometric genus $g$ and degree $d$ in the plane through $3d + g - 1$ points.

The parameter $3d+g-1$ is chosen to be the threshold where the family of such curves becomes finite. A method for computing these invariants classically is described in [6]. Mikhalkin posed and solved the corresponding problem for tropical plane curves. One wrinkle in the tropical case is that one must assign each curve of the given genus and degree through the chosen points a *multiplicity*, which is easiest to describe as the product, over all faces in the Newton subdivision, of twice the area of each face (observe that for smooth curves, all faces are triangles of area $\frac{1}{2}$, so the multiplicity is 1). However, if the curves are counted with these multiplicities, then the tropical analogs of the number $N_{g,d}$ are well-defined, and can be computed. In addition, Mikhalkin proved a correspondence theorem, which demonstrates that the invariants $N_{g,d}$ are equal for the tropical and classical problems. The enumerative method proposed in Mikhalkin proceeds by consider lattice paths in the Newton polygon, and thus gives a concrete, combinatorial approach to this enumerative problem. In addition, the same method applies equally well to Newton polygons other than right triangles, and in this way it provides a method for calculating the analogous invariants for any toric surface.

The Gromov-Witten invariants had already been calculated by classical means. However, the same enumerative techniques also suffice to compute the *Welschinger invariants*, which so far can only be calculated by tropical means. The Welschinger invariants are analogous to the Gromov-Witten invariants in real algebraic geometry, although so far they have only been proved to be well-defined for genus 0 curves. The problem is this: given a parameter $d$ and $3d-1$ general points in $\mathbf{RP}^2$, how many rational curves of degree $d$, defined over $\mathbf{R}$, pass through these points? This question is ill-formed as it stands, because the number depends on the choice of points. However, as the points vary, the solution curves are created and annihilated in pairs, hence the number of solutions is well-defined if they are counted with an appropriate sign (just as intersection theory over $\mathbf{R}$ must keep track of orientation, since there is no natural orientation as for complex manifolds). The correct sign for a solution curve is $(-1)^m$, where $m$ is the number of nodes that occur in the complexification of the curve that appear in the real plane as an isolated point (that is, that are locally $x^2+y^2=0$). Once the curves are counted with these signs, their number becomes invariant; it is called the Welschinger invariant $W_d$.

Mikhalkin devises a means of calculating all Welschinger invariants and proves that it gives the correct number for classical real algebraic geometry. In face, the enumeration method proceeds first by finding all tropical solutions, but counting them with a different multiplicity than in the complex case: if a curve was counted with multiplicity $m$ for the Gromov-Witten invariant $N_{0,d}$, then for the Welschinger invariant $W_d$, it should be counted with multiplicity $(-1)^{(m-1)/2}$ if $m$ is odd, and not counted at all otherwise. Intuitively, if $m$ is even, then the tropical curve found corresponds to an even number of complex curves, which annihilate each other in pairs when included in the Welschinger count.

# 6 Tropical curves via specialization

We now elaborate on the method described in subsection 2.3 for specializing a smooth curve to a tropical curve. Throughout this section, $R$ will be a complete discrete valuation ring with algebraically closed residue field $k$ and field of fractions $K$, and $\bar{K}$ will be a fixed algebraic closure of $K$. We will usually have in mind $R = \mathbf{C}[[t]]$, and $\bar{K}$ the field of Puiseux series, although other choices of $R$ may be used to prove results in arbitrary characteristic.

The key results of this section are the specialization lemma 6.1, which translates between ranks of linear series on tropical and algebraic curves, and theorem 6.7 from deformation theory, which shows the existence of curves specializing to a given graph. In section 6.3 we present a recent application of this method, and in section 6.2 we give the promised justification for our assertion that tropical curves have a god-given canonical divisor within their canonical divisor class.

This entire section closely follows the exposition from [2]. Indeed, the reader is probably better served by consulting Baker's original exposition, but we shall describe it anyway.

## 6.1 The specialization map and specialization lemma

Let $X$ be a smooth curve over $K$, and $\mathfrak{X}$ a regular strongly semistable model with dual graph $G$. We shall denote by $X_{\bar{K}}$ the curve $X \times_{\mathrm{Spec} K} \mathrm{Spec}\bar{K}$, and by $\Gamma$ the metric graph associated to $G$ by assigning all edges unit length. Our first objective is the define the specialization map:

$$\tau_* : \mathrm{Div}(X_{\bar{K}}) \to \mathrm{Div}(\Gamma),$$

We prove that $\tau_*$ preserves degree and effective divisors, and then prove the specialization lemma:

**Lemma 6.1** (Specialization lemma, [2])**.** *For any divisor $A$ on $X_{\bar{K}}$,*

$$r(A) \leq r(\tau_*(A)). \tag{6}$$

We begin by defining a map $\rho : \mathrm{Div}(X) \to \mathrm{Div}(G)$, as follows. Because $X$ is smooth and $\mathfrak{X}$ is regular, Weil divisors are the same as Cartier divisors, hence we can make use of the intersection theory of arithmetic surfaces, which is described in [17].

**Definition 6.2.** Let $C_1, \ldots, C_n$ be the irreducible components of the special fiber, corresponding to vertices $v_1, \ldots, v_n$ in $G$. For any divisor $A \in \mathrm{Div}(X)$, let $\mathcal{A}$ be the closure of $A$ in $\mathfrak{X}$. Then:

$$\rho(A) = \sum_{i=1}^{n} (C_i.A) v_i. \tag{7}$$

The map $\rho$ is degree-preserving, preserves linear equivalence, and preserves effective divisors. We omit the details, which follow from intersection theory on arithmetic surfaces; see [2] for further discussion.

In order to construct a map from the divisor group of $X_{\bar{K}}$, we must understand the behavior of this apparatus under field extension. The key facts are the following two lemmas, whose proofs we omit; further details can be found in [2], which in turn relegates the technical facts about dual graphs and base change to [7].

**Lemma 6.3.** *Suppose that $\mathfrak{X}$ is totally degenerate. For a finite extension $K'$ of $K$, with $R'$ the integral closure of $R$ in $K'$, there is a unique relatively minimal totally degenerate strongly semistable model $\mathfrak{X}'$ dominating the fiber product $\mathfrak{X} \times_{SpecR} SpecR'$. The dual graph $G'$ of the special fiber $\mathfrak{X}'_k$ is given by subdividing each edge of $G$ into a path of length $e$, where $e$ is the ramification index of $K'/K$.*

Hence, we may regard both $G$ and $G'$ as giving rise to the same metric graph $\Gamma$, if the edges of $G'$ are taken to have length $1/e$. This suggests that we should be able to define a single map to $\mathrm{Div}(\Gamma)$, valid for all field extensions. This is almost true.

**Lemma 6.4.** *The map $\rho' : Div(X') \to Div(\Gamma)$ agrees with $\rho$ for divisors which are finite sums of points defined over $K$. For an arbitrary divisor $A$ on $X$ corresponding to divisor $A'$ on $X$, $\rho(A)$ and $\rho'(A')$ are linearly equivalent on $\Gamma$.*

To get some idea why $\rho$ and $\rho'$ do not agree on all divisors, recall that the divisors of $X$ can be identified with $\mathrm{Gal}(K'/K)$-invariant divisors on $X'$. Hence in passing to a field extension, a single divisor may split into a sum of Galois-conjugate divisors, whose images in $\mathrm{Div}(\Gamma)$ may be supported on points that are not vertices of $G$, but are created in $G'$ by the subdivision. Forunately, the result is the same up to linear equivalence.

Given this apparatus, we are able to define the map $\tau_*$.

**Definition 6.5.** Let $\tau : X_{\bar{K}} \to \Gamma$ send a point $p$ defined over some extension $K'$ of $K$ to the vertex of $\Gamma$ corresponding to the intersection of the closure of $p$ in $\mathfrak{X}'$ with the special fiber. Let $\tau_*$ be the induced map $\mathrm{Div}(X_{\bar{K}}) \to \mathrm{Div}(\Gamma)$.

Now that we have established the definition of the specialization map, one technical detail is needed in order to prove the specialization lemma. The issue is that the image of $\tau_*$ is not all of $\mathrm{Div}(\Gamma)$; it is only those divisors supported on points of rational distance from the vertices. Fortunately, this does not change things.

**Lemma 6.6.** *Let $\Gamma$ be a metric graph with integer edge lengths. For $A \in Div(\Gamma)$ supported only on rational points (points of rational distance to all vertices), let $r_{\mathbf{Q}}(A)$ be defined the same way as $r(A)$, except that we only subtract effective divisors supported on rational points. Then $r_{\mathbf{Q}}(A) = r(A)$.*

*Proof.* This is corollary 1.5 from [2], which refers to a rational approximation argument found in [12]. □

*Proof of the specialization lemma.* Suppose that $r(\tau_*(A)) < n$. Then there exists an effective divisor $E$ of degree $n$, supported on rational points of $\Gamma$, such that $\tau_*(A) - E$ is not equivalent to any effective divisor. Then there exists an effective divisor $F$ of degree $n$ on $X$ such that $\tau_*(F) = E$. But then $A - F$ cannot be equivalent to any effective divisor, since otherwise its image under $\tau_*$ would be equivalent to an effective divisor. Thus $r(A) < n$ as well. The result follows. □

In practice, it is difficult to actually construct models $\mathfrak{X}$. Fortunately, for many applications one does not have to: the following lemma from deformation theory ensures the existence of models specializing to any chosen graph.

**Theorem 6.7.** *For any connected graph $G$ and complete discrete valuation ring $R$ with infinite residue field $k$, there exists a curve $X$ with totally degenerate strongly semistable regular model $\mathfrak{X}$ over $R$ whose special fiber has dual graph $G$.*

*Proof.* This is the subject of appendix B to [2]. □

The specialization lemma and theorem 6.7 constitute a technical apparatus that allows theorems about algebraic curves to be proved by purely combinatorial means. We shall see an example in section 6.3.

## 6.2 The canonical divisor of a graph is canonical

The definition of the canonical divisor of a tropical curve in section 3 was entirely unmotivated, aside from how conveniently it served as a dualizer in the Riemann-Roch theorem. However, we certainly must wonder whether the divisor itself (and not just its class) is truly canonical. However, specialization provides a succinct explanation for why we should prefer this particular divisor.

**Proposition 6.8.** *Let $\mathfrak{X}$ be a totally degenerate strongly semistable regular model for a curve $X$, and let $G$ be the dual graph of the special fiber. Then for any canonical divisor $\mathfrak{K}$ for $\mathfrak{X}$, the image of the restriction of $\mathfrak{K}$ to $X$ under the specialization map is the canonical divisor of $\Gamma$, as defined in section 3.*

*Proof.* The proof is a straightforward intersection theory calculation. Let $C_1, \dots, C_n$ be the irreducible components of $\mathfrak{X}_k$, corresponding to vertices $v_1, \dots, v_n$ in $G$. Then by the adjunction formula for arithmetic surfaces (see [17]),

$$\begin{aligned} \mathcal{L}(K_{C_i}) &= \mathcal{L}(\mathfrak{K} + C_i)|C_i \\ \deg(K_{C_i}) &= \mathfrak{K}.C_i + C_i.C_i \end{aligned}$$

Now, $\deg(K_{C_i}) = -2$ since $C_i$ has genus 0. To calculate $C_i.C_i$ observe that $C_i.(C_1 + \cdots + C_n) = 0$ since the latter is a fiber of the family and the former is contained entirely in a fiber. Also, $C_i.C_j$ is 1 if there is an edge between $v_i$ and $v_j$, and 0 otherwise. Hence $C_i.C_i = -\text{val}(v_i)$. Therefore, by definition 6.5, the coefficient of $v_i$ in the specialization of $\mathfrak{K}$ to $G$ is $\mathfrak{K}.C_i = \text{val}(v_i) - 2$, and we obtain definition 4.4 of the canonical divisor of a graph (or tropical curve).

□

It is worth noting that it is not the case that every canonical divisor on $X$ specializes to the canonical divisor on $G$ (although it must always specialize to a divisor in the canonical class). This is because a canonical divisor on $X$ may not have a canonical divisor on $\mathfrak{X}$ as its closure.

We briefly remark that one intriguing consequence of this is that while ranks of divisors may jump during specialization (Baker provides several examples in [2], involving modular curves, demonstrating that the inequality in the specialization lemma can be strict), the ranks of divisors summing to a canonical divisor will always jump by the same amount, since the Riemann-Roch theorem holds both for the curve $X$ and the tropical curve $\Gamma$.

## 6.3 A tropical proof of the Brill-Noether theorem

We conclude this article with a brief description of a recent and exciting application of tropical techniques to classical geometry: it has recently produced a new proof of the Brill-Noether theorem in all characteristics. This theorem was originally prove by Griffiths and Harris [13].

**Theorem 6.9.** *Let $X$ be an algebraic curve of genus $g$. For integers $r, d$, let $W_d^r(X)$ be the scheme of divisor classes in $Pic_d(X)$ which move in a linear series of dimension at least $r$. Let $\rho = g - (r+1)(g-d+r)$. Then if $X$ is a general curve:*

1. *If $\rho < 0$, then $W_d^r(X)$ is empty.*
2. *If $\rho < 0$, then the dimension of $W_d^r(X)$ is $\min(\rho, g)$.*

The fact that $\min(\rho, g)$ is a lower bound for the dimension of $W_d^r(X)$ is not difficult, and has been known for a long time. That $g$ is an upper bound is trivial, since $g$ is the dimension of the Jacobian. Thus, due to semicontinuity of $W_d^r$ as $X$ varies and the irredicibility of the moduli space of curves, the difficult aspect of the theorem is showing existence. In particular, it is necessary to show that for $\rho < 0$, there exists some curve $X$ with no $r$-dimensional degree $d$ linear series for part 1. For part 2, if $\rho \geq 0$, it sufficed to show the existence of a curve $X$ and an effective divisor $E$ of degree $r + \rho + 1$ on $X$, such that there are no linear series of rank $r$ and degree $d$ containing $E$. The existence of both of these types of curves will prove the Brill-Noether theorem.

Theorem 6.7 and the specialization lemma reduce this existence problem to pure combinatorics: if a metric graph $\Gamma$ with integer side lengths can be constructed such that it does not have any linear series of the type described above, then, in all characteristics, we can be assured that some curve degenerates to $\Gamma$ via specialization, and hence proves the necessary existence statement.

In a recent paper [8], Cools, Draisma, Payne, and Robeva construct such a family of metric graphs. The graphs they consider have $g+1$ vertices $v_0, \ldots, v_g$, and $2g$ edges: on of length $l_i$ and one of lenght $m_i$ between vertices $v_{i-1}$ and $v_i$. The main result of [8] is that for the lengths $l_i, m_i$ chosen away from some collection of hyperplanes, $\Gamma$ does not support and linear series of the types that we wish to exclude. What is remarkable about this paper is that the argument is entirely combinatorial, once one takes for granted the admittedly technical black boxes of specialization and deformation theory.

An intriguing question remains in this line of study, namely of the enumerative problem for $\rho = 0$. By the Brill-Noether theorem, the number of linear systems of rank $r$ and degree $d$ is finite when $\rho = 0$, and the number of such systems has been calculated. One can ask the analogous question for metric graphs of the family consider in [8], and the same number is obtained. However, unlike in the case of the Gromov-Witten and Welschinger calculations in [19] (discussed in section 5.6), it is not known whether this equality is an accident. Put differently, it is not known whether there is a bijection between thus systems on the curve and the graph, or if this equality is purely a coincidence due to all relevant terms canceling (see [8], conjecture 1.5).

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