Below are all of the additional practice problems suggested on problem sets 6 to 10 (those covered on midterm 2). These are all useful for exam review. I will put at least one of these problems (perhaps with minor modifications) on the exam.

1. \((3.1.3)\) Order of Elements in \(S_n\). Prove Proposition \[3.4\]. In other words, prove that the order of an element \(\sigma \in S_n\) is the least common multiple of the lengths of the cycles in its cycle decomposition.

2. \((4.4.16)\) Find the number of conjugacy classes of \(S_4\) and the number of elements in each of these classes.

3. \((5.1.1)\) Let \(G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}\) be the direct product of \((\mathbb{Z}/4\mathbb{Z}, +)\) and \((\mathbb{Z}/3\mathbb{Z}, +)\), and let \(H = \langle (2,0) \rangle\) be a subgroup of \(G\). Find the right cosets of \(H\) in \(G\).

4. \((4.3.8)\) Let \(G = S_4\), and let \(V = \mathbb{R}^4\). In Problem \[7\] we defined an action of \(G\) on \(V\).
   
   (a) For this action, what is the stabilizer of \((3, \sqrt{2}, 3, \sqrt{2})\)? Find a familiar group that is isomorphic to this stabilizer.
   
   (b) For \(g \in G\), let \(W(g) = \{ v \in V \mid g \cdot v = v \}\). Prove that \(W(g)\) is a subspace of \(V\). Find a basis for \(W(g)\) when \(g = (1 \ 3) \in S_4\).

5. \((4.4.13)\) Let \(G = S_3\) and \(V = \mathbb{R}^3\). In Problem \[7\] we defined an action of \(G\) on \(V\). (Also see Problem \[4\]) What is the orbit of \((3, \sqrt{2}, 3)\) in this action? What about \((4, 4, 4)\)? What are the possible orbit sizes for this action?

6. \((5.2.2)\) If \(G\) is a noncyclic group of order 27, then for how many elements \(x\) of \(G\) do we have \(x^9 = e\)?

7. \((4.1.4)\) Let \(G\) be a subgroup of \(S_n\). Hence every element of \(G\) is a permutation of \([n] = \{1, \ldots, n\}\). Let \(e_i\) be the element of \(\mathbb{R}^n\) with a 1 in the \(i\)th coordinate and zeros in all other coordinates. The set \(B = \{e_1, \ldots, e_n\}\) is the standard basis for \(\mathbb{R}^n\). Define an action of \(G\) on \(B\) by
   
   \[\sigma \cdot e_i = e_{\sigma(i)}\]

   Extend this action to an action of \(G\) on \(\mathbb{R}^n\) as follows: If \(v \in \mathbb{R}^n\) then, for some scalars \(\alpha_1, \ldots, \alpha_n\), we have \(v = \alpha_1 e_1 + \cdots + \alpha_n e_n\). For \(\sigma \in G\), we define
   
   \[\sigma \cdot v = \alpha_1 e_{\sigma(1)} + \alpha_2 e_{\sigma(2)} + \cdots + \alpha_n e_{\sigma(n)}\].

   (a) Let \(n = 3\), let \(G = S_3\), and let \(v = (\sqrt{2}, -8, 4) \in \mathbb{R}^3\). Find \(\sigma \cdot v\) and \(\tau \cdot v\), where \(\sigma = (1 \ 2 \ 3)\) and \(\tau = (2 \ 3)\).

   (b) Show that the above definition does indeed give an action of \(G\) on \(\mathbb{R}^n\).

   (c) Can you generalize the above action to an action of any subgroup of \(S_n\) on any \(n\) dimensional vector space with a designated basis?
8. \((5.1.6)\) Let \(G\) be a group, and let \(H \leq G\) with \(|G : H| = 2\).

(a) If \(K\) is a subgroup of \(G\) with at least one element not in \(H\). Show that \(G = HK\).

(b) Is it possible to find \(y \in G\) such that \(yH \neq Hy\)?

9. \((5.1.10)\) Let \(G\) be a group, and let \(H \leq G\). Recall Definition 4.24 of \(N_G(H)\), the normalizer of \(H\) in \(G\). Show that
\[N_G(H) = \{x \in G \mid xHx^{-1} = H\} = \{x \in G \mid xH = Hx\}.
\]

10. \((5.1.2)\) Let \(G = (\mathbb{Z}, +)\) be the group of integers, and let \(H = (5\mathbb{Z}, +)\) be the subgroup of \(G\) consisting of all multiples of 5. Describe the right cosets of \(H\) in \(G\).

11. \((5.2.5)\) Let \(D_{10} = \langle a, b \mid a^5 = b^2 = e, ba = a^4b \rangle\) be the dihedral group of order 10. Assume \(x\) and \(y\) are two distinct elements of order two in \(D_{10}\). Let \(H = \langle x, y \rangle\). What can you say about \(|H|\)? Can \(x\) and \(y\) commute? Give your reasons.

12. \((10.1.1)\) Find all normal subgroups of \(D_8\) and of \(S_3\).

13. \((10.1.9)\) Find a group \(G\), with subgroups \(H\) and \(K\), such that \(H \triangleleft K\), \(K \triangleleft G\), but \(H\) not normal in \(G\).

14. \((10.2.4)\) If \(M\) and \(N\) are normal subgroups of a group \(G\), show that \(M \cap N\) is also a normal subgroup of \(G\).

15. \((10.3.3)\) Let \(G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}\) and let \(H\) be the subgroup of \(G\) generated by \((2, 2)\).

(a) What are the elements of \(H\)?

(b) What are the elements of \(G/H\)?

(c) Find a familiar group that is isomorphic to \(G/H\).

16. \((10.3.9)\) Let \(G\) be a group and let \(N \triangleleft G\). Assume that \(|G : N| = m\). Let \(x \in G\). Prove that \(x^m \in N\).

17. \((11.3.5)\) Let \(D_8\) and \(S_3\), as usual, be the dihedral group of order 8 and the symmetric group of degree 3 respectively. Assume \(\phi : D_8 \to S_3\) is a homomorphism. What are the possibilities for \(|\ker(\phi)|\) and \(|\text{Im}(\phi)|\)? For each possibility, give an explicit example.

18. \((15.1.3)\) Let \(d\) be an integer (positive or negative) not divisible by a square of a prime, and \(\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}\). Let \(N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}\) be defined by \(N(a + b\sqrt{d}) = a^2 - db^2\). Prove that, for \(x, y \in \mathbb{Z}[\sqrt{d}]\), we have
\[N(xy) = N(x)N(y)\].

19. (15.1.5) Show that, without using ±1 as one of the factors, neither 3 nor \(2 + \sqrt{5}i\) can be factored in \(\mathbb{Z}[\sqrt{5}i]\).

20. (15.2.6) Is \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) a field? Is \(\mathbb{Z}/4\mathbb{Z}\) a field? Can you find a field with four elements? If so, give its addition and multiplication tables explicitly.

21. (15.2.7) Let \(F_2 = (\mathbb{Z}/2\mathbb{Z}, +, \cdot)\) and define \(E = \{ \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \mid a, b \in F_2 \}\). How many elements does \(E\) have? With the usual matrix addition and multiplication, is \(E\) a field?

22. (15.2.11) Let \(X\) be a non-empty set, and recall (Definition 2.20) that \(2^X\) is the set of all subsets of \(X\), and for \(A\) and \(B\) subsets of \(X\), their symmetric difference is denoted by \(\triangle\) and is defined by

\[ A \triangle B = (A - B) \cup (B - A). \]

Show that \((2^X, \triangle, \cap)\) is a commutative ring with identity. Is it an integral domain?

23. (15.2.13) Find the group of units of \(\mathbb{Z}/5\mathbb{Z}\), \(\mathbb{Z}/6\mathbb{Z}\), \(\mathbb{Z}/12\mathbb{Z}\), and \(\mathbb{Z}/24\mathbb{Z}\).

24. (16.1.1) If \(D\) is an integral domain and \(R\) a subring of \(D\) with at least two elements, then is \(R\) necessarily an integral domain? Either prove that it is, or give an example where it is not.

25. (16.1.4) **Proof of Theorem 16.12d and 16.12e.** Let \(R\) and \(S\) be rings, and \(\phi: R \to S\) a ring homomorphism. Let \(R'\) and \(S'\) be subrings, respectively, of \(R\) and \(S\). Prove that \(\phi(R')\) and \(\phi^{-1}(S')\) are subrings, respectively, of \(S\) and \(R\).

26. (16.1.7) Let \(R\) be a ring with identity. How many ring homomorphisms \(\phi: \mathbb{Z} \to R\) are there with \(\phi(1) = 1_R\)?

27. (16.1.10) Let \(R = \mathbb{Q}[\sqrt{2}]\) and \(S = \mathbb{Q}[\sqrt{3}]\). Show that the only ring homomorphism from \(R\) to \(S\) is the trivial one. In particular, conclude that \(R\) and \(S\) are not isomorphic rings. In other words, assume \(f: R \to S\) is a ring homomorphism. Show that \(f(r) = 0\) for all \(r \in R\).

28. (16.1.20) Let \(R\) be a ring with identity, and let \(J\) be an ideal of \(R\). Assume that \(J\) contains a unit of \(R\). Prove that \(J = R\).