• **Read:** §12 (you may skip Theorem 12.7) and §13 up to the first sentence of page 124.

• **Suggestion:** Work (or think about) the following problems. Problems marked with a * have answers given at the back of the book.
  - §12: 1*, 4*
  - §13: 1*, 2

0. (Not for real points) Tell me your best group-theory-themed Halloween costume idea.

1. Let $\phi : G \rightarrow H$ be a group homomorphism.
   - (a) Prove that if $\phi$ is injective and $H$ is a abelian, then $G$ is abelian.
   - (b) Prove that if $\phi$ is surjective and $G$ is abelian, then $H$ is abelian.
   - (c) Prove that if $\phi$ is surjective and $G$ is cyclic, then $H$ is cyclic.

2. Let $G$ be a group, and define a function $\phi : G \rightarrow G$ by $\phi(g) = g^{-1}$.
   - (a) Prove that $\phi$ is bijective.
   - (b) Prove that if $G$ is abelian, then $\phi$ is an isomorphism.
   - (c) Prove that if $G$ is not abelian, then $\phi$ is not a homomorphism.

   **Terminology:** An isomorphism from a group to itself (i.e. a bijective homomorphism $\phi : G \rightarrow G$) is called an **automorphism**.

3. Let $G$ be a cyclic group of order $n$, and $k$ an integer such that $(k, n) = 1$. Prove that the function $\phi : G \rightarrow G$ given by $\phi(g) = g^k$ is an automorphism (see the definition above).

   **Hint:** make use of Problem 10 of problem set 5.

4. Let $a$ be an element of a group $G$. Define a function $\phi_a : G \rightarrow G$ by $\phi_a(x) = axa^{-1}$.
   - (a) Prove that $\phi_a$ is an automorphism.
   - (b) Consider to set of elements $a \in G$ such that $\phi_a$ equal to the identity function (i.e. $\phi_a(x) = x$ for all $x$). What set is this (this set has come up in other contexts, and has a name)?

5. Suppose that $\phi : G \rightarrow H$ is a surjective group homomorphism, and that both $G$ and $H$ are finite groups. Prove that $|H|$ divides $|G|$.

6. Let $G$ be an abelian group and let $D$ be the subset of $G \times G$ consisting of elements of the form $(g, g)$. Prove that $D \triangleleft (G \times G)$, and that $(G \times G)/D \cong G$ (use the fundamental theorem 13.2).

7. Let $G$ denote the set of all $2 \times 2$ matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, with $a, c \neq 0$ (these are called “upper triangular” matrices). This is a subgroup of $GL(2, \mathbb{R})$ (you can assume this without proof). Let $H$ denote the subset of $G$ consisting of all elements in which $a = c = 1$.
   - (a) Prove that $H$ is a subgroup of $G$. 

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due Wednesday 10/31 by 10pm.
(b) Use the fundamental theorem (13.2) to prove that

\[ G/H \cong (\mathbb{R} - \{0\}, \cdot) \times (\mathbb{R} - \{0\}, \cdot). \]

8. Suppose that \( G \) is a group of odd order, and \( n \geq 2 \) is an integer. Let \( \phi : S_n \to G \) be a group homomorphism (where \( S_n \) is the symmetric group of degree \( n \)).

(a) Prove that any transposition (2-cycle) is in \( \ker \phi \).
(b) Prove that \( \phi \) must be the trivial homomorphism (i.e. \( \phi(x) = e_G \) for all \( x \in S_n \)).