Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

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Grading - For Instructor Use Only

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This page intentionally left blank. You may use it for scratchwork.
1. [8 points] Let \( G, H \) be two groups. Prove that \( G \times H \) is isomorphic to \( H \times G \).

Define \( \varphi : G \times H \to H \times G \) by
\[
\varphi((g,h)) = (h,g).
\]

Observe that
\[
\begin{align*}
\varphi((g_1,h_1)(g_2,h_2)) &= \varphi((g_1,g_2,h_1,h_2)) \\
&= (h_1,h_2, g_1,g_2) \\
&= (h_1,g_1)(h_2,g_2) \\
&= \varphi((g_1,h_1))\varphi((g_2,h_2)).
\end{align*}
\]

So \( \varphi \) is a (group) homomorphism.

\( \varphi \) is surjective, since \( \forall (h,g) \in H \times G, \ (h,g) = \varphi((g,h)) \).

\( \varphi \) is injective, since
\[
\begin{align*}
\varphi((a_1,h_1)) = \varphi((a_2,h_2)) \iff (h_1,a_1)(h_2,a_2) \\
&\iff (h_1,h_2) = (a_2,a_1) \\
&\iff h_1 = h_2 \text{ and } a_1 = a_2 \\
&\iff (a_1,h_1) = (a_2,h_2).
\end{align*}
\]

So \( \varphi \) is an isomorphism, and \( G \times H \cong H \times G \).
2. [8 points] Prove that if $G$ is a cyclic group, then there exists a surjective group homomorphism $\phi : \mathbb{Z} \to G$.

Let $g$ be a generator of $G$.

Define $\varepsilon : \mathbb{Z} \to G$ by $\varepsilon(n) = g^n$.

This is a group homomorphism since $\forall m, n \in \mathbb{Z}$,

\[ \varepsilon(m+n) = g^{m+n} = g^m g^n = \varepsilon(m) \varepsilon(n). \]

$\varepsilon$ is surjective since $\forall g' \in G$, $\exists n \in \mathbb{Z}$ with $g' = g^n$ (defn of "generator"),

so $g' = \varepsilon(n) \in \text{im} \phi$. 
3. [8 points] Let $R$ be a ring, and $a \in R$ an element.

(a) Prove that if $a$ is not a zero-divisor, and $b, c \in R$ satisfy $ab = ac$, then $b = c$.

\[
\text{If } ab = ac, \text{ then } \frac{ab - ac = ac - ac}{= 0_R} \Rightarrow a(b - c) = 0_R.
\]

Since $a \text{ isn't a zero divisor}$, it follows that $b - c = 0_R$.

hence $b - c + c = 0_R + c$ \implies $b = c$,

as desired.

(b) Prove that if $a$ is a zero-divisor, then there exist two elements $b, c \in R$ with $b \neq c$ but $ab = ac$.

Since $a$ is a zero divisor, \exists $b \in R$ st. $b \neq 0_R$ & $ab = 0_R$.

Let $c = 0_R$.

Then $ab = 0_R = a \cdot 0_R = ac$,

but $b \neq c$ since we assumed that $b \neq 0_R$. 


4. [8 points] Suppose that $G$ is an abelian group, and let $H$ be the set of all elements of $G$ with finite order.

(a) Prove that $H$ is a normal subgroup of $G$.

Subgroup:

- **Closed under multiplication:** if $a, b \in H$, then $\exists m, n \in \mathbb{Z}^+ \text{ st } a^m = b^n = e_G$. Hence since $G$ is abelian,
  \[(ab)^{mn} = a^{mn} b^{mn} = (e_G)^n (e_G)^m = e_G,
  \]
  so $o(ab) | mn \Rightarrow o(ab) < \infty \Rightarrow ab \in H$.

- **Closed under inversion:** if $a \in H$, then $\exists n \in \mathbb{Z}^+ \text{ st } a^n = e_G$.

  Then $(a^{-1})^n = (a^{-1})^{-1} = e_G = e_G$.

So $a^{-1} \in H$ as well.

Normal:

If $a \in H$ and $g \in G$, then $\exists n \in \mathbb{Z}^+ \text{ st } a^n = e_G$.

So $(gag^{-1})^n = gag^{-1} gag^{-1} g \cdots gag^{-1} = ga^n g^{-1} = ge_a g^{-1} = e_G$.

So $gag^{-1} \in H$ as well.

$\Rightarrow$ $H$ is normal in $G$.

(b) Prove that all elements of $G/H$ besides the identity have infinite order.

Consider any element $Hg \in G/H$, and suppose it has finite order. Then $\exists n \in \mathbb{Z}^+ \text{ st.}

(Hg)^n = e_{G/H} = He_G.

$\Rightarrow$ $Hg^n = He_G$

$\Rightarrow$ $g^n \in H$

$\Rightarrow$ $\exists m \in \mathbb{Z}^+ \text{ st. } (g^n)^m = e_G$

$\Rightarrow$ $g^{nm} = e_G$

$\Rightarrow$ $o(g) < \infty$.

So in fact $g$ itself is in $H$, i.e. $Hg = He_G$.

So the only element of $G/H$ of finite order is the identity element of $G/H$. 
5. [8 points] Let \( \phi : R \to S \) be a surjective ring homomorphism.

(a) Define \( \ker \phi \).

\[
\ker \phi = \{ x \in R : \phi(x) = 0_S \}.
\]

(b) Prove that \( \ker \phi \) is an ideal of \( R \).

- **Nonemptiness**: \( \phi(0_R) = 0_S \), so \( 0_R \in \ker \phi \). So \( \ker \phi \neq \emptyset \).
- **Closure under \(-\)**: if \( x, y \in \ker \phi \), then
  \[
  \phi(x-y) = \phi(x) - \phi(y) = 0_S - 0_S = 0_S,
  \]
  so \( x - y \in \ker \phi \).
- **Closure under \(*\)**: if \( x \in \ker \phi \) & \( a \in R \), then
  \[
  \phi(ax) = \phi(a)\phi(x) = \phi(a) \cdot 0_S = 0_S
  \]
  & \( \phi(xa) = \phi(x)\phi(a) = 0_S \cdot \phi(a) = 0_S \),
  so \( ax \) & \( xa \) are in \( \ker \phi \) as well.

(c) Prove that if \( S \) is a field, then \( \ker \phi \) is a *maximal* ideal of \( R \).

- By the fund. thm. of ring homs., since \( \phi \) is surjective,
  \[
  S \cong R/\ker \phi.
  \]

We proved in clan that if \( R \) is comm. w/ unity, that an ideal \( I \subset R \) is max if \( R/I \) is a field.
Hence, since \( S \) is a field & \( R/\ker \phi \cong S \), hence
must be a maximal ideal.