1. [12 points] Let $G$ be a group, and $N$ a normal subgroup such that $[G : N] = m$. Prove that for all $g \in G$, $g^m \in N$.

2. [12 points] Let $R$ denote the ring of all $2 \times 2$ matrices with real entries. Let $S$ denote the following subset of $R$.

$$S = \left\{ \begin{pmatrix} a & -5b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$$

(a) Prove that $S$ is a subring of $R$.

(b) What element is the additive identity $0_S$? Does $S$ have a multiplicative identity $1_S$?

3. [12 points] Let $G$ be a finite group, and suppose that we have an action of $G$ on a set $\Omega$.

(a) Suppose $\alpha \in \Omega$. Define the stabilizer $\text{Stab}_G(\alpha)$ of $\alpha$, and prove that it is a subgroup of $G$.

(b) Suppose that $|G| = 27$ and $|\Omega| = 10$. Prove that there exists at least one element $\alpha \in \Omega$ such that $\text{Stab}_G(\alpha) = G$.

Hint: use the fundamental counting principle.

4. [12 points] Let $R$ be a commutative ring with unity. Suppose that $a \in R$ satisfies $a^2 = a$ and $a \neq 0_R, 1_R$.

(a) Prove that $a$ is a zero-divisor.

(b) Define $\langle a \rangle = \{ar : r \in R\}$ and $\langle 1_R - a \rangle = \{(1_R - a)r : r \in R\}$ as usual. Prove that the map $\phi : R \to \langle a \rangle$ given by $\phi(r) = ar$ is a ring homomorphism. Carefully identify any places in your argument where you use the assumption that $R$ is commutative.

(c) Prove that $R/ \langle 1_R - a \rangle$ and $\langle a \rangle$ are isomorphic rings.

Suggestion: first prove that $\ker \phi = \langle 1_R - a \rangle$. 