Below are all of the additional practice problems suggested on problem sets 6 to 10 (those covered on midterm 2). These are all useful for exam review. I will put at least one of these problems (perhaps with minor modifications) on the exam.

1. (**3.1.3**) **Order of Elements in S_n.** Prove Proposition 3.4. In other words, prove that the order of an element $\sigma \in S_n$ is the least common multiple of the lengths of the cycles in its cycle decomposition.

2. (**4.4.16**) Find the number of conjugacy classes of S_4 and the number of elements in each of these classes.

3. (**5.1.1**) Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ be the direct product of $(\mathbb{Z}/4\mathbb{Z}, +)$ and $(\mathbb{Z}/3\mathbb{Z}, +)$, and let $H = \langle(2, 0)\rangle$ be a subgroup of G. Find the right cosets of H in G.

4. (**4.3.8**) Let $G = S_4$, and let $V = \mathbb{R}^4$. In Problem 7, we defined an action of G on V.
   (a) For this action, what is the stabilizer of $(3, \sqrt{2}, 3, \sqrt{2})$? Find a familiar group that is isomorphic to this stabilizer.
   (b) For $g \in G$, let $W(g) = \{v \in V \mid g \cdot v = v\}$. Prove that $W(g)$ is a subspace of V. Find a basis for $W(g)$ when $g = (1\ 3) \in S_4$.

5. (**4.4.13**) Let $G = S_3$ and $V = \mathbb{R}^3$. In Problem 7, we defined an action of G on V. (Also see Problem 4.) What is the orbit of $(3, \sqrt{2}, 3)$ in this action? What about $(4, 4, 4)$? What are the possible orbit sizes for this action?

6. (**5.2.2**) If G is a noncyclic group of order 27, then for how many elements x of G do we have $x^9 = e$?

7. (**4.1.4**) Let G be a subgroup of S_n. Hence every element of G is a permutation of $[n] = \{1, \dots, n\}$. Let e_i be the element of $\mathbb{R}^n$ with a 1 in the ith coordinate and zeros in all other coordinates. The set $B = \{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Define an action of G on B by
$$\sigma \cdot e_i = e_{\sigma(i)}.$$
Extend this action to an action of G on $\mathbb{R}^n$ as follows: If $v \in \mathbb{R}^n$ then, for some scalars α_1, ..., α_n, we have $v = \alpha_1 e_1 + \cdots + \alpha_n e_n$. For $\sigma \in G$, we define
$$\sigma \cdot v = \alpha_1 e_{\sigma(1)} + \alpha_2 e_{\sigma(2)} + \cdots + \alpha_n e_{\sigma(n)}.$$
   (a) Let $n = 3$, let $G = S_3$, and let $v = (\sqrt{2}, -8, 4) \in \mathbb{R}^3$. Find $\sigma \cdot v$ and $\tau \cdot v$, where $\sigma = (1\ 2\ 3)$ and $\tau = (2\ 3)$.
   (b) Show that the above definition does indeed give an action of G on $\mathbb{R}^n$.
   (c) Can you generalize the above action to an action of any subgroup of S_n on any n dimensional vector space with a designated basis?

8. (***5.1.6***) Let G be a group, and let $H \leq G$ with $|G : H| = 2$.
   (a) If K is a subgroup of G with at least one element not in H. Show that $G = HK$.
   (b) Is it possible to find $y \in G$ such that $yH \neq Hy$?

9. (**5.1.10**) Let G be a group, and let $H \leq G$. Recall Definition 4.24 of $\mathbf{N}_G(H)$, the normalizer of H in G. Show that
$$\mathbf{N}_G(H) = \{x \in G \mid xHx^{-1} = H\} = \{x \in G \mid xH = Hx\}.$$

10. (**5.1.2**) Let $G = (\mathbb{Z}, +)$ be the group of integers, and let $H = (5\mathbb{Z}, +)$ be the subgroup of G consisting of all multiples of 5. Describe the right cosets of H in G.

11. (**5.2.5**) Let $D_{10} = \langle a, b \mid a^5 = b^2 = e, ba = a^4 b\rangle$ be the dihedral group of order 10. Assume x and y are two distinct elements of order two in D_{10}. Let $H = \langle x, y\rangle$. What can you say about $|H|$? Can x and y commute? Give your reasons.

12. (**10.1.1**) Find all normal subgroups of D_8 and of S_3.

13. (**10.1.9**) Find a group G, with subgroups H and K, such that $H \lhd K$, $K \lhd G$, but H not normal in G.

14. (***10.2.4***) If M and N are normal subgroups of a group G, show that $M \cap N$ is also a normal subgroup of G.

15. (**10.3.3**) Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and let H be the subgroup of G generated by $(2, 2)$.
   (a) What are the elements of H?
   (b) What are the elements of G/H?
   (c) Find a familiar group that is isomorphic to G/H.

16. (***10.3.9***) Let G be a group and let $N \lhd G$. Assume that $|G : N| = m$. Let $x \in G$. Prove that $x^m \in N$.

17. (**11.3.5**) Let D_8 and S_3, as usual, be the dihedral group of order 8 and the symmetric group of degree 3 respectively. Assume $\phi : D_8 \to S_3$ is a homomorphism. What are the possibilities for $|\ker(\phi)|$ and $|\text{Im}(\phi)|$? For each possibility, give an explicit example.

18. (***15.1.3***) Let d be an integer (positive or negative) not divisible by a square of a prime, and $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$. Let $N \colon \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$ be defined by $N(a + b\sqrt{d}) = a^2 - db^2$. Prove that, for $x, y \in \mathbb{Z}[\sqrt{d}]$, we have
$$N(xy) = N(x)N(y).$$

19. (**15.1.5**) Show that, without using ± 1 as one of the factors, neither 3 nor $2+\sqrt{5}i$ can be factored in $\mathbb{Z}[\sqrt{5}i]$.

20. (**15.2.6**) Is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ a field? Is $\mathbb{Z}/4\mathbb{Z}$ a field? Can you find a field with four elements? If so, give its addition and multiplication tables explicitly.

21. (**15.2.7**) Let $\mathbb{F}_2 = (\mathbb{Z}/2\mathbb{Z}, +, \cdot)$ and define $E = \{\begin{bmatrix} a & b \\ b & a+b \end{bmatrix} \mid a, b \in \mathbb{F}_2\}$. How many elements does E have? With the usual matrix addition and multiplication, is E a field?

22. (**15.2.11**) Let X be a non-empty set, and recall (Definition 2.20) that 2^X is the set of all subsets of X, and for A and B subsets of X, their *symmetric difference* is denoted by $\triangle$ and is defined by
$$A \triangle B = (A - B) \cup (B - A).$$
Show that $(2^X, \triangle, \cap)$ is a commutative ring with identity. Is it an integral domain?

23. (**15.2.13**) Find the group of units of $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$, and $\mathbb{Z}/24\mathbb{Z}$.

24. (**16.1.1**) If D is an integral domain and R a subring of D with at least two elements, then is R necessarily an integral domain? Either prove that it is, or give an example where it is not.

25. (**16.1.4**) **Proof of Theorem 16.12d and 16.12e.** Let R and S be rings, and $\phi\colon R \to S$ a ring homomorphism. Let R' and S' be subrings, respectively, of R and S. Prove that $\phi(R')$ and $\phi^{-1}(S')$ are subrings, respectively, of S and R.

26. (**16.1.7**) Let R be a ring with identity. How many ring homomorphisms $\phi\colon \mathbb{Z} \to R$ are there with $\phi(1) = 1_R$?

27. (**16.1.10**) Let $R = \mathbb{Q}[\sqrt{2}]$ and $S = \mathbb{Q}[\sqrt{3}]$. Show that the only ring homomorphism from R to S is the trivial one. In particular, conclude that R and S are not isomorphic rings. In other words, assume $f : R \to S$ is a ring homomorphism. Show that $f(r) = 0$ for all $r \in R$.

28. (***16.1.20***) Let R be a ring with identity, and let J be an ideal of R. Assume that J contains a unit of R. Prove that $J = R$.