Below are all of the additional practice problems suggested on problem sets throughout the semester. At least one of these problems will appear on the final exam (possibly with minor revision), to ensure that at least one problem is something that you've seen before.

1. (**1.1.3**) List the symmetries of an isosceles triangle.

2. (**1.1.4**)
   (a) List the symmetries of a rectangle.
   (b) Write the multiplication table for the symmetries of a rectangle.

3. (**1.2.2**) Let $\Omega = \mathbb{Z}$ be the set of integers. Define $f : \Omega \to \Omega$ by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}.$$

   Is $f \in \mathrm{Perm}(\Omega)$? If so, what is its inverse? What is f^2? What about f^3?

4. (**1.2.5**) Construct a complete multiplication table for S_3. What is the center (see Definition 1.7) of S_3? If $f = (1\ 2\ 3)$, what is $\mathbf{C}_{S_3}(f)$, the centralizer of f in S_3?

5. (**1.3.1**)
   (a) Find $-\frac{3}{4} - 4$ in $\mathbb{Z}/7\mathbb{Z}$.
   (b) In $\mathbb{Z}/12\mathbb{Z}$ does every non-zero element have a multiplicative inverse (i.e., for $a \in \mathbb{Z}/12\mathbb{Z}$ can we find b such that $ab = 1$)?
   (c) In $\mathbb{Z}/7\mathbb{Z}$ does every non-zero element have a multiplicative inverse?
   (d) We want to know for which integers $n > 1$ every non-zero element of $\mathbb{Z}/n\mathbb{Z}$ has a multiplicative inverse. Look at some examples and make a conjecture. You do not have to prove your conjecture.

   Comment: The textbook sometimes writes (as above) "$\frac{a}{b}$ in $\mathbb{Z}/n\mathbb{Z}$" as a shorthand for ab^{-1}. I usually avoid this notation since it has the potential to cause confusion.

6. (**1.3.2**) Consider the addition operation on $\mathbb{Z}/7\mathbb{Z}$. Start with the element $a = 3$ and find $2a = a+a$, $3a = a+a+a$, and so on until at least $20a$. Do you notice a pattern? Now change a to 4 and repeat what you did. Make a general conjecture based on the patterns that you found. Repeat what you did for $\mathbb{Z}/6\mathbb{Z}$. Is there any difference?

7. (**1.4.4**) How many elements does $\mathrm{GL}(2,3)$ have? Justify your answer without an appeal to Theorem 1.64. Can you extend your argument to $\mathrm{GL}(2,p)$ where p is an arbitrary prime?

8. (**1.4.6**) List the elements of $\mathrm{SL}(2,2)$? What are the possible values for a determinant of a matrix over $\mathbb{Z}/2\mathbb{Z}$? What can you say about the relationship between $\mathrm{GL}(n,2)$ and $\mathrm{SL}(n,2)$?

9. (**2.1.1**) Let I_n be the $n \times n$ identity matrix. Is
$$\{rI_n \mid r > 0, r \in \mathbb{R}\}$$
a group under matrix multiplication?

10. (**2.1.3**) Let $\mathbb{Z}$ denote the set of integers, and let
$$G = \{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a \in \mathbb{Z}\}.$$
Prove that G together with the usual matrix multiplication forms a group.

11. (**2.1.6**) Let n be a positive integer. For which n is S_n abelian? Prove your assertion.

12. (**2.2.3**) If G is a group in which $a^2 = e$ for all $a \in G$, show that G is abelian.

13. (**2.3.4**) Find the order of each of the elements of the group $((\mathbb{Z}/8\mathbb{Z})^{\times}, \cdot)$. Is this group cyclic? Do the same for the group $((\mathbb{Z}/10\mathbb{Z})^{\times}, \cdot)$.

14. (**2.3.9**) Let ℓ be an integer greater than 1, and let G be a finite group with no element of order ℓ. Can there exist $a \in G$ with $\ell \mid o(a)$? Prove your assertion.

15. (***2.3.16***) Let G be a group and let $x, y \in G$. Assume that $xy = yx$, $o(x) = p$, and $o(y) = q$, where p and q are distinct prime numbers. What can you say about $o(xy)$?

16. (**2.3.21**) Consider a fixed shuffle of a deck of cards. Does the repeating of this fixed shuffle some finite (positive) number of times bring the deck eventually back to its original order? Why?

17. (**2.4.3**) Are the groups $(\mathbb{Z}/12\mathbb{Z}, +)$ and $(\mathbb{Z}/13\mathbb{Z})^{\times}$ isomorphic?

18. (***2.5.7***)
    (a) Let m and n be integers greater than 1. What is the order of the element $(1, 1)$ in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$? Make a conjecture.
    (b) Under what conditions would $(1, 1)$ be a generator for $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$?

19. (**2.4.16**) Give an example of two groups G and H, an element $x \in G$, and a homomorphism $\phi : G \to H$ such that $o(x)$ does not equal $o(\phi(x))$.

20. (**2.5.11**) Assume that $G \times H$ is an abelian group. Can we conclude that G and H are abelian?

21. (**2.6.2**) Let $G = (\mathbb{Z}/12\mathbb{Z}, +)$. Find all subgroups of G.

22. (**2.6.3**) Find all subgroups of $(\mathbb{Z}/18\mathbb{Z}, +)$.

23. (**2.6.9**) Let G be a group, and assume that a and b are two elements of order 2 in G. If $ab = ba$, then what can you say about $\langle a, b \rangle$?

24. (**3.1.1**) Let $\sigma = (a_1\ a_2\ \cdots\ a_m) \in S_n$. Find σ^{-1}.

25. (**3.1.4**) What is the smallest positive integer n for which S_n has an element of order 15? What about an element of order 11?

26. (**3.1.5**) Does S_7 have a subgroup isomorphic to $\mathbb{Z}/12\mathbb{Z}$? Either prove that it does not, or exhibit such a subgroup.

27. (**3.2.2**) Let x and y be two three-cycles. Can xy be a four-cycle? Either give an example, or prove that it is impossible.

28. (**3.2.3**) Define $\phi : S_n \to \mathbb{Z}/2\mathbb{Z}$ by

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is an even permutation,} \\ 1 & \text{if } x \text{ is an odd permutation.} \end{cases}.$$

Show that ϕ is a group homomorphism.

29. (**3.2.5**) The alternating group A_6 has how many elements of order 3?

30. (**3.2.8**) Is A_4 isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$? Why?

31. (**3.1.3**) **Order of Elements in S_n.** Prove Proposition 3.4. In other words, prove that the order of an element $\sigma \in S_n$ is the least common multiple of the lengths of the cycles in its cycle decomposition.

32. (**4.4.16**) Find the number of conjugacy classes of S_4 and the number of elements in each of these classes.

33. (**5.1.1**) Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ be the direct product of $(\mathbb{Z}/4\mathbb{Z}, +)$ and $(\mathbb{Z}/3\mathbb{Z}, +)$, and let $H = \langle (2, 0) \rangle$ be a subgroup of G. Find the right cosets of H in G.

34. (**4.3.8**) Let $G = S_4$, and let $V = \mathbb{R}^4$. In Problem 37, we defined an action of G on V.

    (a) For this action, what is the stabilizer of $(3, \sqrt{2}, 3, \sqrt{2})$? Find a familiar group that is isomorphic to this stabilizer.

    (b) For $g \in G$, let $W(g) = \{v \in V \mid g \cdot v = v\}$. Prove that $W(g)$ is a subspace of V. Find a basis for $W(g)$ when $g = (1\ 3) \in S_4$.

35. (**4.4.13**) Let $G = S_3$ and $V = \mathbb{R}^3$. In Problem 37, we defined an action of G on V. (Also see Problem 34.) What is the orbit of $(3, \sqrt{2}, 3)$ in this action? What about $(4, 4, 4)$? What are the possible orbit sizes for this action?

36. (**5.2.2**) If G is a noncyclic group of order 27, then for how many elements x of G do we have $x^9 = e$?

37. (**4.1.4**) Let G be a subgroup of S_n. Hence every element of G is a permutation of $[n] = \{1, \ldots, n\}$. Let e_i be the element of $\mathbb{R}^n$ with a 1 in the ith coordinate and zeros in all other coordinates. The set $B = \{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Define an action of G on B by

    $$\sigma \cdot e_i = e_{\sigma(i)}.$$

    Extend this action to an action of G on $\mathbb{R}^n$ as follows: If $v \in \mathbb{R}^n$ then, for some scalars α_1, …, α_n, we have $v = \alpha_1 e_1 + \cdots + \alpha_n e_n$. For $\sigma \in G$, we define

    $$\sigma \cdot v = \alpha_1 e_{\sigma(1)} + \alpha_2 e_{\sigma(2)} + \cdots + \alpha_n e_{\sigma(n)}.$$

    (a) Let $n = 3$, let $G = S_3$, and let $v = (\sqrt{2}, -8, 4) \in \mathbb{R}^3$. Find $\sigma \cdot v$ and $\tau \cdot v$, where $\sigma = (1\ 2\ 3)$ and $\tau = (2\ 3)$.

    (b) Show that the above definition does indeed give an action of G on $\mathbb{R}^n$.

    (c) Can you generalize the above action to an action of any subgroup of S_n on any n dimensional vector space with a designated basis?

38. (***5.1.6***) Let G be a group, and let $H \leq G$ with $|G : H| = 2$.

    (a) If K is a subgroup of G with at least one element not in H. Show that $G = HK$.

    (b) Is it possible to find $y \in G$ such that $yH \neq Hy$?

39. (**5.1.10**) Let G be a group, and let $H \leq G$. Recall Definition 4.24 of $\mathbf{N}_G(H)$, the normalizer of H in G. Show that

    $$\mathbf{N}_G(H) = \{x \in G \mid xHx^{-1} = H\} = \{x \in G \mid xH = Hx\}.$$

40. (**5.1.2**) Let $G = (\mathbb{Z}, +)$ be the group of integers, and let $H = (5\mathbb{Z}, +)$ be the subgroup of G consisting of all multiples of 5. Describe the right cosets of H in G.

41. (**5.2.5**) Let $D_{10} = \langle a, b \mid a^5 = b^2 = e, ba = a^4b \rangle$ be the dihedral group of order 10. Assume x and y are two distinct elements of order two in D_{10}. Let $H = \langle x, y \rangle$. What can you say about $|H|$? Can x and y commute? Give your reasons.

42. (**10.1.1**) Find all normal subgroups of D_8 and of S_3.

43. (**10.1.9**) Find a group G, with subgroups H and K, such that $H \lhd K$, $K \lhd G$, but H not normal in G.

44. (***10.2.4***) If M and N are normal subgroups of a group G, show that $M \cap N$ is also a normal subgroup of G.

45. (**10.3.3**) Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and let H be the subgroup of G generated by $(2, 2)$.
    (a) What are the elements of H?
    (b) What are the elements of G/H?
    (c) Find a familiar group that is isomorphic to G/H.

46. (***10.3.9***) Let G be a group and let $N \lhd G$. Assume that $|G : N| = m$. Let $x \in G$. Prove that $x^m \in N$.

47. (**11.3.5**) Let D_8 and S_3, as usual, be the dihedral group of order 8 and the symmetric group of degree 3 respectively. Assume $\phi : D_8 \to S_3$ is a homomorphism. What are the possibilities for $|\ker(\phi)|$ and $|\mathrm{Im}(\phi)|$? For each possibility, give an explicit example.

48. (***15.1.3***) Let d be an integer (positive or negative) not divisible by a square of a prime, and $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$. Let $N \colon \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$ be defined by $N(a + b\sqrt{d}) = a^2 - db^2$. Prove that, for $x, y \in \mathbb{Z}[\sqrt{d}]$, we have

$$N(xy) = N(x)N(y).$$

49. (**15.1.5**) Show that, without using ± 1 as one of the factors, neither 3 nor $2 + \sqrt{5}i$ can be factored in $\mathbb{Z}[\sqrt{5}i]$.

50. (**15.2.6**) Is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ a field? Is $\mathbb{Z}/4\mathbb{Z}$ a field? Can you find a field with four elements? If so, give its addition and multiplication tables explicitly.

51. (**15.2.7**) Let $\mathbb{F}_2 = (\mathbb{Z}/2\mathbb{Z}, +, \cdot)$ and define $E = \{\begin{bmatrix} a & b \\ b & a+b \end{bmatrix} \mid a, b \in \mathbb{F}_2\}$. How many elements does E have? With the usual matrix addition and multiplication, is E a field?

52. (**15.2.11**) Let X be a non-empty set, and recall (Definition 2.20) that 2^X is the set of all subsets of X, and for A and B subsets of X, their *symmetric difference* is denoted by $\triangle$ and is defined by
$$A \triangle B = (A - B) \cup (B - A).$$
Show that $(2^X, \triangle, \cap)$ is a commutative ring with identity. Is it an integral domain?

53. (**15.2.13**) Find the group of units of $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$, and $\mathbb{Z}/24\mathbb{Z}$.

54. (**16.1.1**) If D is an integral domain and R a subring of D with at least two elements, then is R necessarily an integral domain? Either prove that it is, or give an example where it is not.

55. (**16.1.4**) **Proof of Theorem 16.12d and 16.12e.** Let R and S be rings, and $\phi\colon R \to S$ a ring homomorphism. Let R' and S' be subrings, respectively, of R and S. Prove that $\phi(R')$ and $\phi^{-1}(S')$ are subrings, respectively, of S and R.

56. (**16.1.7**) Let R be a ring with identity. How many ring homomorphisms $\phi\colon \mathbb{Z} \to R$ are there with $\phi(1) = 1_R$?

57. (**16.1.10**) Let $R = \mathbb{Q}[\sqrt{2}]$ and $S = \mathbb{Q}[\sqrt{3}]$. Show that the only ring homomorphism from R to S is the trivial one. In particular, conclude that R and S are not isomorphic rings. In other words, assume $f : R \to S$ is a ring homomorphism. Show that $f(r) = 0$ for all $r \in R$.

58. (***16.1.20***) Let R be a ring with identity, and let J be an ideal of R. Assume that J contains a unit of R. Prove that $J = R$.

59. (**16.2.11**) Let
$$\mathbb{Z}_{(2)} = \{r \in \mathbb{Q} \mid r = \frac{a}{b}, \text{ with } a, b \in \mathbb{Z}, \ \gcd(a,b) = 1, \text{ and } b \text{ odd}\}.$$
In other words, any rational number whose denominator, when written in reduced form, is odd is in $\mathbb{Z}_{(2)}$. The operations for $\mathbb{Z}_{(2)}$ are the usual addition and multiplication of rational numbers.
   (a) Is $\mathbb{Z}_{(2)}$ a ring? An integral domain? A field?
   (b) What are the units of $\mathbb{Z}_{(2)}$?
   (c) What is $\langle 3 \rangle$? What is $\langle \frac{1}{3} \rangle$? What is $\langle 2 \rangle$?
   (d) Can you find a maximal ideal in $\mathbb{Z}_{(2)}$? Give a proof that the ideal that you are suggesting is actually maximal?
   (e) Can you identify $\mathbb{Z}_{(2)}/\langle 2 \rangle$?

60. (**18.1.1**) Let R be a commutative ring with identity. Assume that u and v are both units in the ring R. Are u and v necessarily associates? What are the associates of zero?

61. (**18.1.5**) Let R be a commutative ring with identity. Show that R is an integral domain if and only if $\{0\}$ is a prime ideal.

62. (**18.1.24**) Let $I = \{\, (2m, 3n) \mid m, n \in \mathbb{Z} \,\}$. Verify that I is an ideal in $\mathbb{Z} \times \mathbb{Z}$. Is this ideal principal, prime, and/or maximal?