1. (**2.3.1**) Find the order of each of the elements of the following groups: $\mathbb{Z}/12\mathbb{Z}$, S_3, and $\mathrm{GL}(2,2)$.

2. (**2.3.6**) Find all the generators of the following cyclic groups:

$$(\mathbb{Z}/6\mathbb{Z},+),((\mathbb{Z}/5\mathbb{Z})^\times,\cdot),(2\mathbb{Z},+),((\mathbb{Z}/11\mathbb{Z})^\times,\cdot).$$

   Comment: the phrase "all of the" is important here! A cyclic group almost never has only one generator. Make sure you find all of them in each case.

3. (***2.3.8***) **Proof of Proposition 2.45.** Complete the proof of Proposition 2.45: If x is an element of order m in a group G, and if, for a positive integer s, we have $x^s = e$, then m divides s.

4. (***2.3.13***) Let G be a group, and $x, y \in G$. Show that $o(yxy^{-1}) = o(x)$.

   Note: recall that problem numbers in italics mean that there is a either an answer or solution in the back of the book. You should consult it, but you should wait until you've solved the problem or gotten stuck, and you must carefully write the solution yourself (not copy the book's solution verbatim).

5. (**2.3.15**) Give an example of a group G and elements $x, y \in G$ with $o(xy) < \min(o(x), o(y))$. Give an example of a group G and elements $x, y \in G$ with $o(xy) > o(x)o(y)$.

6. (**2.3.22**) We have a deck consisting of 10 cards. What element of S_{10} corresponds to the perfect riffle shuffle of this deck? What is the order of this element?

   (See the discussion above this problem in the text for a description of riffle shuffles.)

7. (**2.4.1**) The groups $\mathbb{Z}/6\mathbb{Z}$, S_3, $\mathrm{GL}(2,2)$, and D_6 (the symmetries of an equilateral triangle) are all groups of order 6. Which ones are isomorphic? If two of the groups are isomorphic, give the relabeling explicitly. If two of the groups are not isomorphic then give a reason.

8. (**2.4.6**) Recall that $G = \mathrm{GL}(2,\mathbb{R})$ is the group of all invertible 2×2 matrices with entries in the reals $\mathbb{R}$.

   (a) We want to see if inside G there is a group isomorphic to D_8. In other words, can we find a group isomorphic to D_8 consisting of 2×2 invertible matrices with real entries? Let S be the square in $\mathbb{R}^2$ with vertices at $(\pm 1, \pm 1)$. Think of elements of D_8 as linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. For example a is the linear transformation that rotates every vector $90°$. Find a 2×2 matrix for each of these linear transformations. Do they form a group isomorphic to D_8?

   (b) Can you find a group isomorphic to D_8 consisting of elements of $\mathrm{GL}(2,\mathbb{C})$?

   (c) Can you find a group isomorphic to D_8 consisting of elements of $\mathrm{GL}(2,3)$?

9. (**2.5.6**) Is $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$? Why or why not?

10. (**Bonus problem:** This is meant primarily for fun, if you are interested. It is worth a very small amount of extra credit on the problem set.)

    Let $\star$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x \star z = y$. (This z may depend on x and y.) Prove that $(S, \star)$ is an abelian group.

Some other good problems to try for additional practice (but not to hand in):
2.3.4, 2.3.9, 2.3.16, 2.3.21 2.4.3, 2.5.7