Note: to give you more flexibility in allocating your time now that the final projects are out, the following policy will be in effect for all remaining problem sets: **you can skip** 20% **of the set and still earn full points**. More precisely: when computing grades, I will reduce all scores above 80% down to 80%, and then divide by 0.8. Of course, you will still receive feedback and scores for all problems you submit, and you should think about all of the problems, since all concern material that may occur on exams. This policy is not reflected in the numbers reported on Gradescope, but it will be applied on the spreadsheet where I compute grades.

1. Let R be a ring, and $I \subseteq R$ an ideal. The ideal I is called a *radical ideal* if it has the property that for all $a \in R$ and $n \in \mathbb{Z}^+$, if $f^n \in I$ then $f \in I$. An element a of a ring R is called *nilpotent* if there exists an integer n such that $a^n = 0_R$.
   (a) Prove that every prime ideal is a radical ideal.
   (b) Prove that I is radical if and only if the quotient R/I has no nilpotent elements besides $I + 0_R$.

2. Let I and J be ideals in a ring R. Define
$$I + J = \{x + y : \ x \in I \text{ and } y \in J\}.$$
   (a) Prove that $I + J$ is an ideal in R.
   (b) Prove that if K is an ideal such that $K \supseteq I$ and $K \supseteq J$, then $K \supseteq I + J$. Together with part (a), this justifies the following slogan: "$I + J$ is the smallest ideal containing both I and J."

3. (**18.1.18**) Show that 5 is not irreducible in $\mathbb{Z}[i]$, the ring of Gaussian integers.

4. (**18.1.32**) Let R be a commutative ring with identity, and let a, $b \in R$. Let $I = \langle a \rangle$ and $J = \langle b \rangle$. Show that $a + J$ is a prime element of R/J if and only if $b + I$ is a prime element of R/I.

   Correction: You should also assume that a does not divide b and b does not divide a.

5. (**18.3.12**) Let R be a UFD. Assume that every non-zero prime ideal of R is maximal. Prove that every prime ideal of R is principal.

6. Suppose that R is a Principal Ideal Domain (PID) and $a \in R$ is a nonzero element ($a \neq 0_R$). Prove that $\langle a \rangle$ is a maximal ideal if and only if a is an irreducible element.

7. Let K be a field, and R be a ring that contains K as a subring ($K \subseteq R$), and let $r \in R$ be any element of R. Define a function $\phi_r : \ K[X] \to R$ by the formula $\phi_r(f(X)) = f(r)$.
   (a) Prove that ϕ_r is a ring homomorphism (it is called an *evaluation homomorphism*).

(b) Suppose that ϕ is surjective but not injective. Prove that there exists a nonzero polynomial $f(X) \in K[X]$ such that $R \cong K[X]/\langle f(X)\rangle$. Furthermore, show that there exists a *unique* such polynomial with leading coefficient equal to 1 (such a polynomial is called "monic").

8. Write each polynomial as a product of irreducible polynomials in the specified polynomial ring.

   (a) $2X^3 + X^2 + 2 \in (\mathbb{Z}/3\mathbb{Z})[X]$

   (b) $X^3 + 3X^2 + X + 4 \in (\mathbb{Z}/5\mathbb{Z})[X]$

   (c) $X^2 + 5 \in (\mathbb{Z}/7\mathbb{Z})[X]$

   (d) $X^4 + X^3 + 2X^2 + X + 2 \in (\mathbb{Z}/3\mathbb{Z})[X]$

   (e) $X^5 + X^2 - X - 1 \in (\mathbb{Z}/2\mathbb{Z})[X]$

9. Let F be the field $\mathbb{Z}/3\mathbb{Z}$, and let I be the ideal $\langle X^2 + 1\rangle$ in $F[X]$. The quotient ring $F[X]/I$ is a field with nine elements (you do not need to prove this on your homework, but I strongly suggest you try to do so on your own). For all n from 1 to 9, write the element $(I + X + 1)^n \in F[X]/I$ in the form $I + a + bX$ for integers $a, b \in \{0, 1, 2\}$ (use the coset criterion!). From your computations, prove that the unit group of $F[X]/I$ is cyclic.

Some other good problems to try for additional practice (but not to hand in): 16.2.11, 18.1.1, 18.1.5, 18.1.24 (more may be added later)