Note: to give you more flexibility in allocating your time now that the final projects are out, the following policy will be in effect for all remaining problem sets: **you can skip** 20% **of the set and still earn full points**. More precisely: when computing grades, I will reduce all scores above 80% down to 80%, and then divide by 0.8. Of course, you will still receive feedback and scores for all problems you submit, and you should think about all of the problems, since all concern material that may occur on exams. This policy is not reflected in the numbers reported on Gradescope, but it will be applied on the spreadsheet where I compute grades.

1. Let R be a ring, and let I be an ideal of R.
   (a) Prove that if R is commutative, then so is R/I.
   (b) Prove that if R has unity, then so does R/I.

Terminology comment: in the following two problems, a ring homomorphism is called *trivial* if it sends everything to 0, and nontrivial otherwise.

2. (**16.1.6**) Let R be a ring with identity, and let D be an integral domain. Let $\phi\colon R \to D$ be a non-trivial ring homomorphism. Show that $\phi(1_R)$ is the identity of D.

3. Suppose that $\phi : \mathbb{Z}[\sqrt{5}] \to \mathbb{R}$ is a nontrivial ring homomorphism ("nontrivial" here, and in the previous problem, means that not everything is sent to 0). Prove that $\phi(\sqrt{5})$ is either $\sqrt{5}$ or $-\sqrt{5}$. Deduce that there are exactly three ring homomorphisms from $\mathbb{Z}[\sqrt{5}]$ to $\mathbb{R}$, and give a formula for each one. (You may assume that the claim in the previous problem is true, whether or not you have solved that problem yet.)

4. (**16.1.22**) Let R be a ring and F a field. Let $\phi : F \to R$ be a ring homomorphism. Assume ϕ is not trivial. In other words, there exists $x \in F$ with $\phi(x) \neq 0$. Show that ϕ must be 1-1.

5. (**16.2.7**) Recall that the Gaussian integers are denoted by $\mathbb{Z}[i]$ and are defined by

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}, i^2 = -1\}.$$

   Let $I = \langle 1 + 3i \rangle$ be the ideal generated by $1 + 3i$ in $\mathbb{Z}[i]$, and define $R = \mathbb{Z}[i]/I$.
   (a) $I + i$ and $I + 3$ are two elements of R. Are they equal? What about $I + 9$ and $I + 1$?
   (b) How many elements does R have?
   (c) Can you find a familiar ring that is isomorphic to R?

6. Consider the following subset of $\mathbb{Z}[\sqrt{-5}]$. I claimed in class that this is a non-principal ideal; the purpose of this problem is to work through a proof of that claim.

$$I = \{2a + (1 + \sqrt{-5})b : \ a, b \in \mathbb{Z}[\sqrt{-5}]\}$$

   (After Friday's class, we'll have an alternative notation for this: $I = \langle 2, 1 + \sqrt{-5} \rangle$.)
   (a) Prove that I is an ideal of $\mathbb{Z}[\sqrt{-5}]$.

(b) Prove that $1 \notin I$.

(c) Suppose that $a \in \mathbb{Z}[\sqrt{-5}]$ is an element such that the principal ideal of a contains I. Prove that $N(a)$ divides both $N(2)$ and $N(1+\sqrt{-5})$.

(d) Prove that I is not a principal ideal (the previous two parts will be useful in your proof).

Some other good problems to try for additional practice (but not to hand in): 16.1.1, 16.1.4, 16.1.7, 16.1.10, 16.1.20