1. [6 points] Let R be a ring, and I an ideal in R. Prove that the quotient ring R/I is commutative if and only if $xy - yx \in I$ for all $x, y \in R$.

2. (a) [4 points] List all the elements of the symmetric group S_3, using notation of your choice.
   (b) [4 points] Which elements from part (a) are in the alternating group A_3?
   (c) [4 points] Let $f = (1\ 2\ 3)$. Determine the centralizer $C_{S_3}(f)$ of f in S_3.

   (Recall that the centralizer of f is the set of all elements of the group that commute with f.)

3. Let $\phi : R \to S$ be a ring homomorphism.
   (a) [4 points] Define the *kernel* of ϕ, denoted $\ker \phi$, and prove that it is an ideal.
   (b) [5 points] Assume that R is a commutative ring with unity, and S is an integeral domain. Prove that either $\ker \phi = R$ or $\ker \phi$ is a *prime* ideal.

   (Recall: An integral domain is a commutative ring with unity with at least two elements and no zero divisors. A prime ideal is an ideal $I \neq R$ such that for all $a, b \in R$, if $ab \in I$ then either $a \in I$ or $b \in I$, or both.)

4. Suppose that G is a finite group, and $g \in G$ is an element of order 9.
   (a) [4 points] Prove that $|G|$ is divisible by 9.
   (b) [5 points] Prove that for all integers n, $g^n = e_G$ if and only if $9 \mid n$.

   Suggestion: For the "only if" direction, use the division algorithm for $\mathbb{Z}$.
   (c) [3 points] Determine $o(g^2)$ and $o(g^3)$.

5. Let $R = \mathbb{Z} \times \mathbb{Z}$, and let $I = \{(2m, 3n) : \ m, n \in \mathbb{Z}\}$.
   (a) [4 points] Prove that I is an ideal in R.
   (b) [2 points] Is I a principal ideal? Briefly justify your answer.
   (c) [2 points] Is I a prime ideal? Briefly justify your answer.
   (d) [2 points] Is I a maximal ideal? Briefly justify your answer.

6. Let G be a group, H a subgroup of G, and g an element of G. Define

$$K = gHg^{-1} = \{ghg^{-1} : \ h \in H\}.$$

   (a) [4 points] Prove that $K \leq G$ (K is a subgroup of G).
   (b) [4 points] Prove that $K \cong H$.

7. Let F be a field, and let $F[X]$ denote the polynomial ring over F.
   (a) [4 points] Prove that $F[X]$ is an integral domain. You may assume that $F[X]$ is a commutative ring with unity, as well as any basic facts about degrees of polynomials proved in class.
   (b) [4 points] Let $I = \langle X^2 + 1 \rangle$ denote the principal ideal generated by X^2+1 in $F[X]$. Prove that every element in the quotient ring $F[X]/I$ is equal to $I + (a + bX)$ for some choice of elements $a, b \in F$.

(c) [4 points] Let I be as in part (b). Prove that if $a, b \in F$ satisfy $a^2 + b^2 \neq 0$, then the element $I + a + bX \in F[X]/I$ is a unit in $F[X]/I$.
Hint: mimic the way that inverses are computed in $\mathbb{C}$ or $\mathbb{Q}[\sqrt{-1}]$.

8. [6 points] Let R be an integral domain. Prove that if $p \in R$ is a prime element, then p is also an irreducible element. In your argument, explicitly identify where you use the assumption that R is an integral domain.

(Recall: An element $p \in R$ is *prime* if it is nonzero, it is not a unit, and for all $a, b \in R$ such that $p \mid ab$, either $p \mid a$ or $p \mid b$. An element $p \in R$ is *irreducible* if it is nonzero, it is not a unit, and for all $a, b \in R$ such that $p = ab$, either a is a unit or b is a unit.)

9. Suppose that G is a group, and H is a subgroup of $Z(G)$.

(a) [4 points] Prove that H is a normal subgroup of G.

(b) [6 points] Suppose that the quotient group G/H is cyclic, with generator Hg. Prove that G is abelian.
Hint: First show every element $x \in G$ is equal to hg^n for some $h \in H$ and integer n.

(c) (**Bonus**; up to 2 points of extra credit. I don't recommend spending time on this unless you've completed the rest of the exam!)

Prove that if G is a group of order p^3, for p a prime number, then $g^p \in Z(G)$ for all $g \in G$.