- **Read:** §12 (you may skip Theorem 12.7) and §13 up to the first sentence of page 124.
- **Suggestion:** Work (or think about) the following problems. Problems marked with a $*$ have answers given at the back of the book.
  - §12 : $1*, 4*$
  - §13 : $1*, 2$

0. (Not for real points) Tell me your best group-theory-themed Halloween costume idea.

1. Let $\phi : G \to H$ be a group homomorphism.
   (a) Prove that if ϕ is injective and H is a abelian, then G is abelian.
   (b) Prove that if ϕ is surjective and G is abelian, then H is abelian.
   (c) Prove that if ϕ is surjective and G is cyclic, then H is cyclic.

2. Let G be a group, and define a function $\phi : G \to G$ by $\phi(g) = g^{-1}$.
   (a) Prove that ϕ is bijective.
   (b) Prove that if G is abelian, then ϕ is an isomorphism.
   (c) Prove that if G is not abelian, then ϕ is not a homomorphism.

   Terminology: An isomorphism from a group to itself (i.e. a bijective homomorphism $\phi : G \to G$) is called an *automorphism*.

3. Let G be a cyclic group of order n, and k an integer such that $(k, n) = 1$. Prove that the function $\phi : G \to G$ given by $\phi(g) = g^k$ is an automorphism (see the definition above).

   Hint: make use of Problem 10 of problem set 5.

4. Let a be an element of a group G. Define a function $\phi_a : G \to G$ by
   $$\phi_a(x) = axa^{-1}.$$
   (a) Prove that ϕ_a is an automorphism.
   (b) Consider to set of elements $a \in G$ such that ϕ_a equal to the identity function (i.e. $\phi_a(x) = x$ for all x). What set is this (this set has come up in other contexts, and has a name)?

5. Suppose that $\phi : G \to H$ is a surjective group homomorphism, and that both G and H are finite groups. Prove that $|H|$ divides $|G|$.

6. Let G be an abelian group and let D be the subset of $G \times G$ consisting of elements of the form (g, g). Prove that $D \triangleleft (G \times G)$, and that $(G \times G)/D \cong G$ (use the fundamental theorem 13.2).

7. Let G denote the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, with $a, c \neq 0$ (these are called "upper triangular" matrices). This is a subgroup of $GL(2, \mathbb{R})$ (you can assume this without proof). Let H denote the subset of G consisting of all elements in which $a = c = 1$.
   (a) Prove that H is a subgroup of G.

(b) Use the fundamental theorem (13.2) to prove that

$$G/H \cong (\mathbb{R} - \{0\}, \cdot) \times (\mathbb{R} - \{0\}, \cdot).$$

8. Suppose that G is a group of odd order, and $n \geq 2$ is an integer. Let $\phi : S_n \to G$ be a group homomorphism (where S_n is the symmetric group of degree n).

   (a) Prove that any transposition (2-cycle) is in $\ker \phi$.

   (b) Prove that ϕ must be the trivial homomorphism (i.e. $\phi(x) = e_G$ for all $x \in S_n$).