- **Read:** §11.
- **Suggestion:** Work (or think about) the following problems. Problems marked with a $*$ have answers given at the back of the book.
  - §11 : 2*, 3, 14*

1. Recall that $SL(2,\mathbb{R})$ denotes the subgroup of $GL(2,\mathbb{R})$ consisting of all 2×2 invertible matrices with determinant 1. Prove that $SL(2,\mathbb{R})$ is normal in $GL(2,\mathbb{R})$.
2. Prove that the intersection of any two normal subgroups in a group G is also a normal subgroup.
3. Suppose that G is a group, and N, H are subgroups with $N \triangleleft G$. Prove that the set
$$NH = \{nh : \; n \in N, h \in H\}$$
is a subgroup of G.
4. Suppose that G is an abelian group, and H is a subgroup. Prove that G/H is abelian.
5. Suppose that G is a group of order 21, and $g, h \in G$ are two elements of G such that $o(g) = 7$, $o(h) = 3$, and $hg = g^2h$. It follows from these assumptions that any element of G can be written uniquely as g^ih^j for some $i \in \mathbb{Z}_7$ and $j \in \mathbb{Z}_3$ (you may assume this without proof, but you may find it interesting to think about why; ask me if you are interested).
   (a) Compute each of the following elements. Express each answer as g^ih^j, where $i \in \mathbb{Z}_7$ and $j \in \mathbb{Z}_3$.
   $$(gh)^2, \; (g^4h^2)(g^3h), \; (g^3h)^{-1}$$
   (b) Prove that $\langle g \rangle$ is a normal subgroup of G, but that $\langle h \rangle$ is not.
   (c) Show that $G/\langle g \rangle$ is a cyclic group of order 3.

   Comment: this description of G is very similar to our description of the dihedral groups D_n. Indeed, both are examples of a construction called a *semi-direct product.*
6. Let G be a group. Given any two elements $a, b \in G$, the *commutator* of a and b, denoted $[a, b]$ is defined to be
$$[a, b] = aba^{-1}b^{-1}.$$
   (a) Prove that G is abelian if and only if for all $a, b \in G$, $[a, b] = e$.
   (b) Prove that for a normal subgroup $H \triangleleft G$, the quotient group G/H is abelian if and only if $[a, b] \in H$ for all $a, b \in G$.
7. Consider the quotient group $C = (\mathbb{R}, +)/(\mathbb{Z}, +)$.
   (a) Prove that for every integer n, there exists an order-n subgroup of C.
   (b) Prove that if $x \in \mathbb{R}$ is irrational, then the element $\mathbb{Z} + x$ of C has order ∞.
   (c) (Bonus; for extra credit) Prove that any finite subgroup of C is cyclic.

   Comment: the notation $\mathbb{Z} + x$ gets a bit clunky to write over and over again. You may prefer to instead use the "overline" notation $\overline{x}$ (as used in Chapter 9) for the coset of x.

8. Let G be a group, and $Z(G)$ denote the center of G (the set of all elements that commute with every element of G). Prove that if $G/Z(G)$ is a cyclic group, then G is abelian.

9. Suppose that G is a group, H is a normal subgroup of G with $|H| = 7$ and $[G : H] = 20$. Lagrange's theorem implies that if $x \in H$, then $x^7 = e$. Prove the converse: if $x \in G$ satisfies $x^7 = e$, then $x \in H$.

   Hint: use the quotient group.