- **Read:** §7 and the beginning of §8, up to the middle of page 70. Note that we are skipping §6 for now.
- **Suggestion:** Work (or think about) the following problems. Problems marked with a $*$ have answers given at the back of the book.
  - §7 : 1*, 4*, 8*

1. Suppose x is an element of a group G.
   (a) Prove that if $o(x)$ is finite, then every negative power of x is equal to some nonnegative power of x (this is why, for finite groups, one can find $\langle x \rangle$ by finding $\{e, x, x^2, \cdots\}$ and not worrying about negative powers at all).
   (b) Prove, on the other hand, that if $o(x) = \infty$, then no negative power of x is equal to a positive power of x.
2. The following statement is false, but it is true if it is revised slightly. Correct the statement and prove it: "If G is a cyclic group of order p, where p is prime, then every element of G is a generator of G."
3. Suppose that G is a finite group, and that the only subgroups of G are $\{e\}$ and G itself. Prove that the order of G is either 1 or a prime number.
4. Suppose that $G = \langle g \rangle$ is a cyclic group of order n. Prove that g^m is a generator of G if and only if $(m, n) = 1$.
5. Suppose that g is an element of a group G. Define the *centralizer* $Z(g)$ of g to be the set of all $x \in G$ that commute with g. In other words,

$$Z(g) = \{x \in G : \; xg = gx\}.$$

   Prove that $Z(g)$ is a subgroup of G.
6. Fix an element a of a group G. Define a function $f : \; G \to G$ by

$$f(x) = axa^{-1}$$

   (this function is called "conjugation by a"). Is f injective (one-to-one)? Is f surjective (onto)?

   Comment: You may recognize this function from linear algebra, where it arises as the way to convert a matrix representation of a linear operator from one basis to another.
7. For each statement, either prove it or provide a counterexample.
   (a) If $f : S \to T$ and $g : T \to U$ are functions such that $g \circ f$ is injective (one-to-one) then both f and g are injective.
   (b) If $f : S \to T$ and $g : T \to U$ are functions such that $g \circ f$ is surjective (onto), then both f and g are surjective.
8. Carry out the indicated multiplications in S_6.
   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 4 & 2 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix}$

Note: this is exercise 8.1 *in Saracino. Check your answer to (b) in the back of the book.*

9. Write each permutation as a product of disjoint cycles, and then as a product of transpositions. Determine whether each permutation is even or odd.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

Note: this is exercise 8.2 *in Saracino. Check your answers to (a) and (c) at the back of the book.*