Read: Saracino, §3 – 5.

- **Suggestion:** Work (or think about) the following problems. Problems marked with a $*$ have answers given at the back of the book.
  - §3: 1*, 7
  - §4 : 1*, 4*, 10*
  - §5 : 1*, 4*, 12*

Note: some of these problems concern material that we may not cover in class until early next week.

1. Suppose that G is a nonempty set and $*$ is an associative binary operation on G. Assume that both the left cancellation law and the right cancellation law hold for $(G, *)$ (statements (i) and (ii) from Theorem 3.6).
   (a) Give an example to show that $(G, *)$ need not be a group (try to tinker around for awhile, but ask me for a hint if you get stuck).
   (b) Prove that if G is assumed to be *finite*, then $(G, *)$ must be a group.
2. Consider the binary operation $a * b = a + b - 2$ on $\mathbb{Z}$.
   (a) Prove that $(\mathbb{Z}, *)$ is a group.
   (b) Is $(\mathbb{Z}, *)$ a *cyclic* group? If so, give a generator.
3. In the group $(\mathbb{Z}_{42}, \oplus)$, find the orders of the elements $1, 2, 3, 4, 5, 6$ and 7.
4. Let $G = \langle x \rangle$ be a cyclic group of order 24. List all elements of G that have order 3.
5. Let G be a group, and $a \in G$. Call a second element $b \in G$ *conjugate to* a if there exists an element $x \in G$ such that $b = xax^{-1}$. Show that if b is conjugate to a, then b has the same order as a.
6. Suppose that G is an abelian group, and $x, y \in G$ are elements of finite order. Prove that if $(o(x), o(y)) = 1$, then $o(xy) = o(x)o(y)$.
7. Suppose that G is an abelian group with exactly 91 elements. Suppose that G has an element of order 7, and also an element of order 13. Prove that G is cyclic. (You may use the result of the exercise 6 in your argument even if you have not solved it yet).

   Comment: We will prove later that if p is a prime number dividing the number of elements in a finite group G, then G necessarily has an element of order p. So the exercise above implies that all abelian groups with 91 elements are cyclic.

Update (9/17): because I didn't cover as much as intended this morning, you may turn in these last three problems with next week's problem set if you wish. I still recommend trying to finish them this week, but I want to make sure you have at least two days to think about them.

8. List all subgroups of the group $(\mathbb{Z}_{42}, \oplus)$.
9. Find all subgroups of Q_8 (the group of unit quaternions).
10. Suppose that H is a nonempty subset of a group G, and that for all $x, y \in H$, the element xy^{-1} is in H. Prove that H is a subgroup of G.