Amherst College
Department of Mathematics and Statistics

MATH 350-01 | MIDTERM 2 PRACTICE | FALL 2018

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	Total
Points:	8	8	8	8	8	40
Score:						

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1. [8 points] Let G, H be two groups. Prove that $G \times H$ is isomorphic to $H \times G$.

Define $\varphi : G\times H \longrightarrow H\times G$ by

$$\varphi((g,h)) = (h,g).$$

Observe that

$$\begin{aligned}\varphi((g_1,h_1)(g_2,h_2)) &= \varphi((g_1g_2, h_1h_2)) \\ &= (h_1h_2, g_1g_2) \\ &= (h_1,g_1)(h_2,g_2) \\ &= \varphi((g_1,h_1))\varphi((g_2,h_2)).\end{aligned}$$

So φ is a (group) homomorphism.

φ is surjective, since $\forall (h,g) \in H\times G$, $(h,g) = \varphi((g,h))$.

φ is injective, since

$$\begin{aligned}\varphi((g_1,h_1)) = \varphi((g_2,h_2)) &\Leftrightarrow (h_1,g_1) = (h_2,g_2) \\ &\Leftrightarrow h_1 = h_2 \ \& \ g_1 = g_2 \\ &\Leftrightarrow (g_1,h_1) = (g_2,h_2).\end{aligned}$$

So φ is an isomorphism, and $G\times H \cong H\times G$.

2. [8 points] Prove that if G is a cyclic group, then there exists a surjective group homomorphism $\phi : \mathbb{Z} \to G$.

Let g be a generator of G.

Define $\varepsilon : \mathbb{Z} \longrightarrow G$ by $\varepsilon(n) = g^n$.

This is a group hom. since $\forall m, n \in \mathbb{Z}$,

$$\varepsilon(m+n) = g^{m+n} = g^m g^n = \varepsilon(m)\,\varepsilon(n).$$

ε is surjective since $\forall g' \in G$, $\exists n \in \mathbb{Z}$ w/ $g' = g^n$ (defn of "generator"),

so $g' = \varepsilon(n) \in \operatorname{im} \varphi$.

3. [8 points]] Let R be a ring, and $a \in R$ an element.

(a) Prove that if a is not a zero-divisor, and $b, c \in R$ satisfiy $ab = ac$, then $b = c$.

If $ab=ac$, then

$$ab - ac = ac - ac$$
$$= 0_R$$
$$\Rightarrow a(b-c) = 0_R.$$

Since a isn't a ZD, it follows that $b-c=0_R$,

hence $b-c+c = 0_R + c$

$$\Rightarrow \underline{b=c},$$

as desired.

(b) Prove that if a *is* a zero-divisor, then there exist two elements $b, c \in R$ with $b \neq c$ but $ab = ac$.

Since a is a ZD, $\exists b \in R$ st. $b \neq 0_R$ & $ab = 0_R$

Let $c = 0_R$.

Then $ab = 0_R = a \cdot 0_R = ac$,

but $b \neq c$ since we assumed that

$b \neq 0_R$.

4. [8 points] Suppose that G is an abelian group, and let H be the set of all elements of G with finite order.

(a) Prove that H is a normal subgroup of G.

Subgroup:

closed under mult: if $a, b \in H$, then $\exists m, n \in \mathbb{Z}^+$ w/ $a^m = b^n = e_G$. Hence since G is abelian,

$$(ab)^{mn} = a^{mn} b^{mn} = (e_G)^n (e_G)^m = e_G,$$

so $o(ab) \mid mn \Rightarrow o(ab) < \infty \Rightarrow ab \in H$.

closure under inverse:

if $a \in H$, then $\exists n \in \mathbb{Z}^+$ st. $a^n = e_G$.

Then $(a^{-1})^n = (a^n)^{-1} = e_G^{-1} = e_G$.

So $a^{-1} \in H$ as well.

Normal:

If $a \in H$ & $g \in G$, then $\exists n \in \mathbb{Z}^+$ st $a^n = e_G$.

So $(gag^{-1})^n = ga\underbrace{g^{-1}g}_{=e_G}a\underbrace{g^{-1}g}_{=e_G} \cdots gag^{-1} = ga^n g^{-1} = g e_G g^{-1} = e_G$.

So $gag^{-1} \in H$ as well.

$\Rightarrow H$ is normal in G.

(b) Prove that all elements of G/H besides the identity have infinite order.

Consider any element $Hg \in G/H$, and suppose it has finite order. Then $\exists n \in \mathbb{Z}^+$ st.

$$(Hg)^n = e_{G/H} = He_G.$$

$$\Rightarrow Hg^n = He_G$$

$$\Rightarrow g^n \in H$$

$$\Rightarrow \exists m \in \mathbb{Z}^+ \text{ st. } (g^n)^m = e_G$$

$$\Rightarrow g^{nm} = e_G$$

$$\Rightarrow o(g) < \infty.$$

So in fact g itself is in H, ie. $Hg = He_G$.

So the only element of G/H of finite order is the identity element of G/H.

5. [8 points] Let $\phi : R \to S$ be a surjective ring homomorphism.

(a) Define $\ker \phi$.

$$\ker\varphi = \{x \in R : \varphi(x) = 0_S\}.$$

(b) Prove that $\ker \phi$ is an ideal of R.

nonemptiness: $\varphi(0_R) = 0_S$, so $0_R \in \ker\varphi$. So $\ker\varphi \neq \emptyset$.

closure under $-$: if $x, y \in \ker\varphi$, then

$$\varphi(x-y) = \varphi(x) - \varphi(y) = 0_S - 0_S = 0_S,$$

so $x - y \in \ker\varphi$.

sticky property: if $x \in \ker\varphi$ & $a \in R$, then

$$\varphi(ax) = \varphi(a)\varphi(x) = \varphi(a)\cdot 0_S = 0_S$$

& $\varphi(xa) = \varphi(x)\varphi(a) = 0_S \cdot \varphi(a) = 0_S$

so ax & xa are in $\ker\varphi$ as well.

(c) Prove that if S is a field, and R is a comm ring w/ unity, then $\ker \phi$ is a *maximal* ideal of R.

By the fund. thm. of ring homs., since φ is surjective

$$S \cong R/\ker\varphi.$$

We proved in class that if R is comm w/ unity, that an ideal $I \subseteq R$ is max'l iff R/I is a field.

Hence, since S is a field & $R/\ker\varphi \cong S$, $\ker\varphi$ must be a max'l ideal.