Math 350: Groups, Rings and Fields

Final Exam (spring 2016)

NAME:

- Attempt problems 1-8. Problems 9 and 10 are optional.
- **Justify all your answers**. (If you are unsure whether or not something requires further justification, you are welcome to ask me.) Please write clearly and legibly, and cross out or erase anything that you do not want graded.
- You may use the course textbook (Ch.0-14, 16-21), your class notes, and old homework and exams. Please clearly identify any theorems or previous results you use.
- No other textbooks, websites, calculators or outside help may be used on this exam.

Recall: for our class, you are allowed a one-page (both sides) notesheet.

- All discussion about this exam is strictly prohibited (including conversations about how easy/hard a question is, and how much progress you have made so far).

Problem Number	Possible Points	Points Earned
1	14	
2	13	
3	13	
4	10	
5	14	
6	14	
7	12	
8	10	
9	3*	
10	3*	
Total	100	

1. (14 points) Let $\sigma = (2 \quad 5 \quad 4)(3 \quad 7)$ and $\tau = (4 \quad 6 \quad 5)(1 \quad 7 \quad 3 \quad 2)$ be elements of S_7.

   (a) Express $\sigma\tau$ as a product of disjoint cycles, and use your answer to find the order of $\sigma\tau$.

   $$(2\ 5\ 4)(3\ 7)(4\ 6\ 5)(1\ 7\ 3\ 2) = \begin{pmatrix} 1&2&3&4&5&6&7 \\ 1&5&7&2&4&6&3 \end{pmatrix}\begin{pmatrix} 1&2&3&4&5&6&7 \\ 7&1&2&6&4&5&3 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6&7 \\ 3&1&5&6&2&4& \end{pmatrix}$$

   $$= (1\ 3\ 5\ 2)(4\ 6)$$

   (b) Express $\sigma\tau$ as a product of transpositions and determine whether it is even or odd.

   (c) What is the order of the element $A_7\sigma * A_7\tau$ in the quotient group S_7/A_7?

   (d) Is there an *odd* permutation of order 7 in S_7? If so, give an example, and if not, explain why.

2. (13 points) Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$ and let

$$S = \left\{ \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} \middle| a, b \in \mathbb{Z} \right\}.$$

Show that R and S are isomorphic rings.

3. (13 points) Let G be the set of all 2×2 matrices with real number entries. Then G is a group under the operation of matrix addition. Let

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, a + d = 0 \right\}.$$

Show that G/H and $(\mathbb{R}, +)$ are isomorphic groups.

4. (10 points) Let $\varphi : R \to T$ be a ring homomorphism that is onto, and let I be an ideal of R. Show that $\varphi(I)$ is an ideal of T, where $\varphi(I) = \{\varphi(a) \mid a \in I\}$.
(This theorem is stated in the book, but it is stated without proof. You are required to explicitly prove all necessary parts of this here, instead of citing the theorem.)

5. (14 points) Define a relation $\mathcal{R}$ on the set $M_{2\times 2}(\mathbb{R})$ of 2×2 real matrices by $A\mathcal{R}B$ for $A, B \in M_{2\times 2}(\mathbb{R})$ if and only if there exists an invertible matrix $P \in \mathrm{GL}(2, \mathbb{R})$ such that $A = PB$.

   (a) Show that $\mathcal{R}$ is an equivalence relation on $M_{2\times 2}(\mathbb{R})$.

   (b) What are the equivalence classes of the identity matrix I_2 and the zero matrix O_2? (Note that these two sets will have orders.)

6. (14 points) Let $F = \mathbb{Z}_7$ and $R = F[X]$.

(a) What are the zero divisors in R?

(b) What are the units in R?

(c) Let $f(X) = X^2 + 5$ and let $I = \big(f(X)\big)$. How many elements are in R/I?

(d) Is R/I a field? Justify your answer.

(e) Is R/I a domain? Justify your answer.

7. (12 points) Write down the orders of the following groups:

~~(a) The group of 5×5 permutation matrices with determinant 1. (Recall that a permutation matrix is a matrix with a single entry equal to 1 in every row and column, and 0 everywhere else.)~~

(b) $D_4/Z(D_4)$, where $Z(D_4)$ is the center of D_4

recall: $Z(D_4) = \{z \in D_4 : xz = zx \ \forall x \in G\}$

(c) $\langle x^6 \rangle$, where $o(x) = 10$

(d) G/H where $G = \mathbb{Z}_{15} \times \mathbb{Z}_4$ and $H = \langle (5, 2) \rangle$

8. (10 points) Let G be a group and $H \triangleleft G$.

(a) Show that if $x \in G$ is an element such that $o(x) = m$, then $o(Hx)$ is finite in G/H and divides m.

(b) Show that if $|G/H| = n$, then $g^n \in H$ for all $g \in G$. (Note that this part is completely independent from part (a).)

9. **Optional Bonus Problem:** (3 points) We know that $\mathbb{Z}_{6011}^{\times}$ is a group under $\odot$ (ie multiplication mod 6011), since 6011 is a prime number. Find the inverse of 1001 in $(\mathbb{Z}_{6011}^{\times}, \odot)$. (Your answer should be an explicit integer between 0 and 6010. Note that this can be computed without a calculator.)

Note: this is tricky if you haven't studied the Euclidean algorithm in §4 (which we didn't discuss in class).

10. **Optional Bonus Problem:** (3 points) Give an example of a group G and elements $x, y \in G$ such that $o(x) = o(y) = 2$, but $o(xy)$ is infinite.