Study guide

- (§1.5) Terminology: *homogeneous* linear system, *trivial solution*.
- (§1.5) Once you know that \( A\vec{x} = \vec{b} \) has one solution, you can find all of the other solutions by solving the homogeneous equation \( A\vec{x} = \vec{0} \). Understand why this is.
- (§1.6) Determinants, interpreted as area/volume expansion factor.
- (§1.6) Techniques to evaluate determinants: formula for 2 \( \times \) 2 case; row-reduction; cofactor expansion.

Textbook problems

- §1.5: 26, 28
- §1.6: 10, 22, 26, 52 (hint for 52: use the result of supplementary problem 4. You may also use the fact that \( \det A = \det A^T \).)

Supplemental problems:

1. Suppose that \( A \) is an \( n \times n \) matrix, \( \vec{u} \) is a nonzero vector, and \( A\vec{u} = \vec{0} \) (here, \( \vec{0} \) denotes the \( n \times 1 \) column vector with all entries equal to 0). Prove that \( A \) is not invertible.

2. (a) Let \( A \) be an \( n \times n \) matrix. Suppose that in every row of \( A \), the entries of that row sum to 0. Prove that \( A \) is not invertible.

   *Hint*: Use the result of problem 1.

   (b) Suppose instead that in every column of \( A \), the entries in that column sum to 0. Deduce from part (a) and a problem from the previous assignment that \( A \) is not invertible.

   *Note*: To clarify the wording: part (a) concerns matrices like
   \[
   \begin{pmatrix}
   1 & 2 & -3 \\
   0 & 1 & -1 \\
   1 & -4 & 3
   \end{pmatrix}
   \]
   (note that in each of the three rows, the numbers sum to 0), while part (b) concerns matrices like
   \[
   \begin{pmatrix}
   1 & 0 & 1 \\
   2 & 1 & -4 \\
   -3 & -1 & 3
   \end{pmatrix}
   \]

3. Suppose that \( A \) is a square matrix.

   (a) Prove that if \( A \) has a column consisting entirely of 0s, then \( \det A = 0 \).

   (b) Prove that if two rows of \( A \) are identical, then \( \det A = 0 \).

4. Suppose that \( A \) is an \( n \times n \) matrix and \( c \) is a constant. Prove that

   \[
   \det (c \cdot A) = c^n \det A.
   \]
5. Let $A$ be an $n \times n$ matrix, and let $B = A - 7I$ (where $I$ is the $n \times n$ identity matrix). Prove that the following two sets are equal:

$$\{ \vec{v} \in \mathbb{R}^n : A\vec{v} = 7\vec{v} \} = \{ \vec{v} \in \mathbb{R}^n : B\vec{v} = \vec{0} \}.$$

(Please ask me or one of the course staff about any of this notation if it is new to you! We’ve used it a bit in class, but it takes some getting used to when you are new to it.)