Proofs of claims from the "review of key concepts" handout

Note: following our in-class convention since §3.3, I'll use +, • instead of ⊕, ⊗ below.

Claim 1 Any subspace W of a vector space V contains 0.

Pf W is closed under scalar multiplication. So for every \( \tilde{w} \in W \), \( 0 \cdot \tilde{w} \in W \) as well. But \( 0 \cdot \tilde{v} = 0 \) for all \( \tilde{v} \in V \).

Claim 2 A list of vectors \( \tilde{v}_1, \ldots, \tilde{v}_n \in V \) is linearly independent if and only if

\[ \forall \tilde{v} \in \text{span}\{\tilde{v}_1, \ldots, \tilde{v}_n\}, \text{ there are unique coefficients } c_1, \ldots, c_n \in \mathbb{R} \text{ s.t. } \tilde{v} = c_1 \tilde{v}_1 + \cdots + c_n \tilde{v}_n. \]

Pf 1) Suppose \( \tilde{v}_1, \ldots, \tilde{v}_n \) are lin. indep. Suppose also that \( \tilde{v} \) is in their span. So \( \exists \) constants \( c_1, \ldots, c_n \) s.t. \( \tilde{v} = c_1 \tilde{v}_1 + \cdots + c_n \tilde{v}_n \).

To see that they are unique, suppose that

\[ \tilde{v} = c'_1 \tilde{v}_1 + \cdots + c'_n \tilde{v}_n \]

for some list of constants \( c'_1, \ldots, c'_n \); we will show that in fact \( c_i = c_i' \); we will show that

Indeed:

\[ c_i' \tilde{v}_i + c_i' \tilde{v}_2 + \cdots + c_i' \tilde{v}_n = c_i \tilde{v}_i + c_i \tilde{v}_2 + \cdots + c_i \tilde{v}_n \]  

\[ \Rightarrow (c_i' \tilde{v}_i - c_i \tilde{v}_i) + (c_i' \tilde{v}_2 - c_i \tilde{v}_2) + \cdots + (c_i' \tilde{v}_n - c_i \tilde{v}_n) = 0 \]  

\[ \Rightarrow (c_i' - c_i) \tilde{v}_1 + (c_i' - c_i) \tilde{v}_2 + \cdots + (c_i' - c_i) \tilde{v}_n = 0 \]  

\[ \Rightarrow c_i' - c_i = c_i' - c_i = \cdots = c_i' - c_n = 0 \quad (\text{by lin. independence}) \]  

\[ \Rightarrow c_i' = c_i, \quad c_i' = c_i, \ldots, c_i' = c_i, \]

as claimed.
2) For the converse, suppose that all elements of the span have unique coefficients. Then \( \overline{0} = 0 \cdot \overline{v}_1 + 0 \cdot \overline{v}_2 + \cdots + 0 \cdot \overline{v}_n \) must be the unique way to write \( \overline{0} \) as a lin. comb. of \( \overline{v}_1, \ldots, \overline{v}_n \). So the only way to write

\[
\overline{0} = c_1 \overline{v}_1 + \cdots + c_n \overline{v}_n
\]

is if \( c_1 = c_2 = \cdots = c_n = 0 \). Hence \( \overline{v}_1, \ldots, \overline{v}_n \) is lin. indep.

Claim 3 If \( B = \{\overline{v}_1, \ldots, \overline{v}_n\} \) is a basis for \( V \), then

\( \forall \overline{v} \in V \), there are unique "coordinates" \( c_1, \ldots, c_n \in \mathbb{R} \)

st. \( \overline{v} = c_1 \overline{v}_1 + \cdots + c_n \overline{v}_n \).

Pf Since \( B \) is a basis, \( \text{span} \{\overline{v}_1, \ldots, \overline{v}_n\} = V \) and

\( \overline{v}_1, \ldots, \overline{v}_n \) are lin. indep. So by claim 2, all \( \overline{v} \in V \)

(= \( \text{span} \{\overline{v}_1, \ldots, \overline{v}_n\} \)) determine unique constants \( c_1, \ldots, c_n \)

st. \( \overline{v} = c_1 \overline{v}_1 + \cdots + c_n \overline{v}_n \).

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Claim 4 \( \{\overline{v}_1, \ldots, \overline{v}_n\} \) is lin. indep. iff no vector in the list is in the span of the others.

Pf 1) Suppose that \( \{\overline{v}_1, \ldots, \overline{v}_n\} \) is lin. indep. Suppose

(for contradiction) that one of them, say \( \overline{v}_i \), is

in the span of the others. This means that

\[
\overline{v}_i = c_1 \overline{v}_1 + \cdots + c_{i-1} \overline{v}_{i-1} + c_{i+1} \overline{v}_{i+1} + \cdots + c_n \overline{v}_n.
\]

for some constants \( c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \).

This implies that

\[
\overline{0} = c_1 \overline{v}_1 + \cdots + c_{i-1} \overline{v}_{i-1} + (c_i) \overline{v}_i + c_{i+1} \overline{v}_{i+1} + \cdots + c_n \overline{v}_n,
\]

contradicting the linear independence of \( \overline{v}_1, \ldots, \overline{v}_n \).

So none of \( \overline{v}_1, \ldots, \overline{v}_n \) is a lin. comb. of the others.
2) Conversely, suppose that no \( \vec{v}_i \) is a lin. comb. of the others.

Suppose (for contradiction) that \( \vec{v}_1, \ldots, \vec{v}_n \) are linearly dependent. Then \( \exists c_1, \ldots, c_n \in \mathbb{R} \), not all 0, s.t.

\[
0 = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n.
\]

Suppose \( c_i \neq 0 \) (we've assumed at least one of the \( c_i \)'s isn't 0, so choose one). Then:

\[
-c_i \vec{v}_i = c_1 \vec{v}_1 + \cdots + c_{i-1} \vec{v}_{i-1} + c_{i+1} \vec{v}_{i+1} + \cdots + c_n \vec{v}_n
\]

(mull. by \(-\frac{1}{c_i}\)) \( \Rightarrow \vec{v}_i = (-\frac{c_1}{c_i}) \vec{v}_1 + (-\frac{c_2}{c_i}) \vec{v}_2 + \cdots + (-\frac{c_{i-1}}{c_i}) \vec{v}_{i-1} + (-\frac{c_{i+1}}{c_i}) \vec{v}_{i+1} + \cdots + (-\frac{c_n}{c_i}) \vec{v}_n
\)

So \( \vec{v}_i \) is a lin. comb. of the others, a contradiction. \( \square \)

So \( \vec{v}_1, \ldots, \vec{v}_n \) are in fact lin. indep.

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**Claim 5** If \( \{\vec{v}_1, \ldots, \vec{v}_n\} \) is lin. dependent, then any \( \vec{v} \) in their span can be written

\[
\vec{v} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n
\]

in infinitely many ways.

**Proof** There are constants \( d_1, \ldots, d_n \in \mathbb{R} \), not all 0, s.t.

\[
0 = d_1 \vec{v}_1 + \cdots + d_n \vec{v}_n.
\]

Hence whenever \( \vec{v} \) can be written

\[
\vec{v} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n,
\]

it follows that \( \forall t \in \mathbb{R} \),

\[
\vec{v} = \vec{v} + t \cdot 0
\]

\[
= (c_1 + t \cdot d_1) \vec{v}_1 + (c_2 + t \cdot d_2) \vec{v}_2 + \cdots + (c_n + t \cdot d_n) \vec{v}_n.
\]
Since not all of $d_1, \ldots, d_n$ are 0, the list of coefficients
\[ c_i + t \cdot d_i, \ldots, c_n + t \cdot d_n \]
is different for every choice of $t$ (more precisely: if $d_i \neq 0$, then $c_i + t \cdot d_i = c_i + t' \cdot d_i$ iff \( t = t' \), so the coeff. $c_i + t \cdot d_i$ is different for every value of $t \in \mathbb{R}$).

So $\vec{v}$ is a lin. comb. of $\vec{v}_i, \ldots, \vec{v}_n$ in $\infty$ ways.

\[ \text{Claim 6} \quad \text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \} = \text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \} \]
iff $\vec{v}_n \in \text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \}$.

**Proof**

1) Suppose $\text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \} = \text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \}$.

Then since $\vec{v}_n \in \text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \}$ (it is $0 \cdot \vec{v}_i + \ldots + 0 \cdot \vec{v}_{n-1} + 1 \cdot \vec{v}_n$),

$\vec{v}_n \in \text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \}$ as well.

2) Suppose that $\vec{v}_n \in \text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \}$; say that

$\vec{v}_n = d_1 \vec{v}_1 + \ldots + d_{n-1} \vec{v}_{n-1}$.

Then $\forall \vec{v} \in \text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \}$,

$\vec{v} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$

for some $c_1, \ldots, c_n \in \mathbb{R}$,

$\Rightarrow \quad \vec{v} = c_1 \vec{v}_1 + \ldots + c_{n-1} \vec{v}_{n-1} + c_n \left( d_1 \vec{v}_1 + \ldots + d_{n-1} \vec{v}_{n-1} \right)$

$= (c_1 + d_1 \cdot c_n) \vec{v}_1 + (c_2 + d_2 \cdot c_n) \vec{v}_2 + \ldots + (c_{n-1} + d_{n-1} \cdot c_n) \vec{v}_{n-1}$

$\Rightarrow \quad \vec{v} \in \text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \}$ as well.

So $\text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \} \subseteq \text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \}$. But also $\text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \} \subseteq \text{span} \{ \vec{v}_i, \ldots, \vec{v}_{n-1} \}$ since $c_1 \vec{v}_1 + \ldots + c_{n-1} \vec{v}_{n-1} = c_1 \vec{v}_1 + \ldots + c_{n-1} \vec{v}_{n-1} + 0 \cdot \vec{v}_n$.

So $\text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \} = \text{span} \{ \vec{v}_i, \ldots, \vec{v}_n \}$, as desired.