Review of some key concepts on vector spaces

- A vector space is a set of objects that can be scaled (by real numbers) and added together, in a way that respects a few algebraic laws (axioms).
  
  → key examples: \( \mathbb{R}^n \), \( M_{m \times n} \), \( P_d \)  
  \( (m \times n \text{ matrix}) \) (polynomials of degree \( d \)).

- A subspace is a subset that's closed under addition & scaling.  
  e.g. subspaces of \( \mathbb{R}^3 \) are:  
  
  → all of \( \mathbb{R}^3 \) (3-dim 1)  
  → planes through the origin (2-dim 1)  
  → lines through the origin (1-dim 1)  
  → the set \( \{0\} \) (0-dim 1)

  □ Why do subspaces always contain \( \vec{0} \)?

- The span of a list \( \vec{v}_1, \ldots, \vec{v}_n \in V \) is the set of all linear combinations  
  \[ c_1 \vec{v}_1 \oplus c_2 \vec{v}_2 \oplus \ldots \oplus c_n \vec{v}_n. \]  
  It is a subspace.

- A list \( \vec{v}_1, \ldots, \vec{v}_n \) is linearly independent if linear combinations  
  \( \vec{v} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_n) \) uniquely determine the coefficients \( c_i \) s.t.  
  \[ \vec{v} = c_1 \vec{v}_1 \oplus \ldots \oplus c_n \vec{v}_n. \]

  □ Convince yourself that this is equivalent to the usual defn, which states that  
  \[ c_1 \vec{v}_1 \oplus \ldots \oplus c_n \vec{v}_n = \vec{0} \quad \text{holds only when} \quad c_1 = c_2 = \ldots = c_n = 0.\]

- A basis is an ordered list \( \vec{v}_1, \ldots, \vec{v}_n \in V \) that is both linearly independent and spans all of \( V \).

  → This means that you can assign unique coordinates to each \( \vec{v} \in V \).

  □ Convince yourself why this is true.
Some review questions

Try to convince yourself on an intuitive level why these things are true. Then think about how you'd write a proof.

- A list \( \vec{v}_1, \ldots, \vec{v}_n \) is linearly independent iff no vector in the list is in the span of the others.

- If \( \vec{v}_1, \ldots, \vec{v}_n \) is linearly dependent, then every \( \vec{v} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} \) can be written as a linear combination
  \[ \vec{v} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n \]
  in infinitely many ways.

- \( \text{span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}, \vec{v}_n\} \)
  iff \( \vec{v}_n \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}\} \).
  (Both statements express a sort of "redundancy" of \( \vec{v}_n \).)