1. **Short answer questions.** No explanations are necessary.
   (a) Give a basis for each of the following vector spaces.
   - $\mathbb{R}^3$
   - $M_{2\times 2}$ (the space of 2×2 matrices)
   - $\mathcal{P}_2$ (the space of polynomials of degree at most 2)

   (b) Suppose that B and B' are two bases for a vector space V, and the change of basis matrix is
   $$[I]_B^{B'} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$
   Let $\vec{v} \in V$ be a vector such that $[\vec{v}]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Determine $[\vec{v}]_{B'}$.

   (c) Find the multiple of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ that is closest to the vector $\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$.

2. Consider the two bases $B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

3. Consider the following three vectors in $\mathbb{R}^4$.
   $$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 3 \\ 6 \\ 1 \\ 6 \end{pmatrix}$$
   Denote by W the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.
   (a) Find a basis of W.
   (b) What is the dimension of W?
   (c) Find an *orthonormal* basis for W. (Recall that a basis is called *orthonormal* if any two vectors in the basis are orthogonal, and each vector has norm equal to 1.)
   (d) What element of W is closest to the vector $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$?

   (In other words, which element $\vec{x}$ of W minimizes $\|\vec{x} - \vec{b}\|$?)

4. Suppose that V is an inner product space, and $\vec{u}, \vec{v}$ are two nonzero vectors in V such that $\langle \vec{u}, \vec{v} \rangle = 0$. Prove, directly from the axioms of an inner product, that
   $$\|\vec{u} + 2\vec{v}\| > \|\vec{u}\|.$$

5. Suppose that $f(x)$ is a continuous function on the interval $[-\pi, \pi]$ that has been measured in a laboratory. You have reason to believe that the function should be approximately equal to

some linear combination of $\sin(x)$ and $\cos(x)$. Using your experimental data, you compute the following integrals.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin x \; dx &= 7 \\ \int_{-\pi}^{\pi} f(x)\cos x \; dx &= 13\end{aligned}$$

You may also use, without proof, the values of the following integrals.

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2 x \; dx &= \pi \\ \int_{-\pi}^{\pi} \sin x \cos x \; dx &= 0 \\ \int_{-\pi}^{\pi} \cos^2 x \; dx &= \pi\end{aligned}$$

(a) Let $V = \mathcal{C}^{(0)}[-\pi, \pi]$ denote the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on V, and identify each of the five integrals above as an inner product $\langle f, g\rangle$ of two elements f, g of V.

(b) Suppose that we choose two constants c_1, c_2 and define a function $g(x)$ as follows:

$$g(x) = c_1 \sin x + c_2 \cos x.$$

Find the values of c_1, c_2 that will ensure that $\int_{-\pi}^{\pi}(f(x) - g(x))^2 \; dx$ is as small as possible.

6. [9 points] Suppose that A is an $m \times n$ matrix, and $\vec{u}, \vec{v}, \vec{w}$ are three vectors in $\mathbb{R}^n$ such that $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is a linearly independent set of vectors. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is also a linearly independent set of vectors.