Study guide

- (§1.5) Terminology: *homogeneous* linear system, *trivial solution.*
- (§1.5) Once you know that $A\vec{x} = \vec{b}$ has one solution, you can find all of the other solutions by solving the homogeneous equation $A\vec{x} = \vec{0}$. Understand why this is.
- (§1.6) Determinants, interpreted as area/volume expansion factor.
- (§1.6) Techniques to evaluate determinants: formula for 2×2 case; row-reduction; cofactor expansion.

Textbook problems

- §1.5: 26, 28
- §1.6: 10, 22, 26, 52 (hint for 52: use the result of supplementary problem 4. You may also use the fact that $\det A = \det A^t$.)

Supplemental problems:

1. Suppose that A is an $n \times n$ matrix, $\vec{u}$ is a nonzero vector, and $A\vec{u} = \vec{0}$ (here, $\vec{0}$ denotes the $n \times 1$ column vector with all entries equal to 0). Prove that A is *not* invertible.

2. (a) Let A be an $n \times n$ matrix. Suppose that in every row of A, the entries of that row sum to 0. Prove that A is not invertible.

   Hint: Use the result of problem 1.

   (b) Suppose instead that *in every column* of A, the entries in that column sum to 0. Deduce from part (a) and a problem from the previous assignment that A is not invertible.

   Note: To clarify the wording: part (a) concerns matrices like $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 1 & -4 & 3 \end{pmatrix}$ (note that in each of the three rows, the numbers sum to 0), while part (b) concerns matrices like $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ -3 & -1 & 3 \end{pmatrix}$.

3. Suppose that A is a square matrix.

   (a) Prove that if A has a column consisting entirely of 0s, then $\det A = 0$.

   (b) Prove that if two rows of A are identical, then $\det A = 0$.

4. Suppose that A is an $n \times n$ matrix and c is a consant. Prove that

$$\det(c \cdot A) = c^n \det A.$$

5. Let A be an $n \times n$ matrix, and let $B = A - 7I$ (where I is the $n \times n$ identity matrix). Prove that the following two sets are equal:

$$\{\vec{v} \in \mathbb{R}^n : \; A\vec{v} = 7\vec{v}\} = \{\vec{v} \in \mathbb{R}^n : \; B\vec{v} = \vec{0}\}.$$

(Please ask me or one of the course staff about any of this notation if it is new to you! We've used it a bit in class, but it takes some getting used to when you are new to it.)