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2019f
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F 9/27
class 15
(climate strike day)

Recall: for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad - bc$

means two things:

1) The area expansion factor of the tranformation $\mathbb{R}^2 \to \mathbb{R}^2$ represented by A. w/ a ± sign telling whether it reverses orientation;

2) It detects whether A is invertible:

$$A^{-1} \text{ exists} \iff \det A \neq 0.$$

Today. we generalize to all dimensions & state two algorithms.

Informal defn: for an n×n matrix A, the determinant of A denoted detA, is the factor by which A expands/contracts volume, when viewed as a transformation $\mathbb{R}^n \to \mathbb{R}^n$. It is given a ± sign according to whether it preserves or reverses orientation.

(this is "informal" since I haven't defined "volume" in n dimensions.)

Fact for square A,
A^{-1} exists
$\Updownarrow$
$\det A \neq 0$.

Handy algebraic facts:
1) $\det AB = \det A \cdot \det B$
2) $\det A^t = \det A$

warning
$\det(A+B) \neq \det A + \det B$.

qu How do you compute ~~A?~~ detA? (for larger than 2×2?)

$\boxed{\text{method 1}}$ row-reduction + keeping score.

Fact! Suppose B is obtained from A by a row operation. Then:

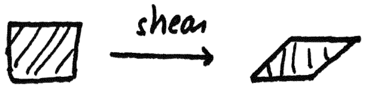
if the row op. was:	then the determinants satisfy:
$R_i \mathrel{+}= c \cdot R_j$	$\det B = \det A$
$R_i \mathrel{*}= c$	$\det B = c \cdot \det A$
$R_i \longleftrightarrow R_j$	$\det B = -\det A$

(side note: "column operations" work the same way!)

Fact 2 $\det I = 1$ (I means "do nothing", so no volume expansion).

geometric reason // just say out loud in class.

$R_i \mathrel{+}= cR_j$ takes all columns of A & "shears" them.



this doesn't change volume.

$R_i \mathrel{*}= c$ magnifies one dim. by c.
This scales volume by $|c|$, & flips orientation if $c<0$.

$R_i \leftrightarrow R_j$ "reflects" all columns across the plane $x_i = x_j$.
This preserves volume, but reverses orientation.

eg.1

$$\det\begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & 2 \\ 2 & 9 & 6 \end{pmatrix} \overset{(R1\leftrightarrow R2)}{=} -\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ \underset{-2}{2} & \underset{-6}{9} & \underset{-4}{6} \end{pmatrix}$$

(R3 -= 2·R1)

$$= -\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 0 & \underset{-3}{3} & \underset{-6/5}{2} \end{pmatrix}$$

($R3 \mathrel{-}= \frac{3}{5}R2$)

$$(*) \quad = -\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & 4/5 \end{pmatrix}$$

a few more += operations clear above the pivots *without* changing the determinant.

$$= -\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4/5 \end{pmatrix}$$

$$\begin{pmatrix} R2 & *=1/5 \\ R3 & *=5/4 \end{pmatrix}$$

$$= -5\cdot\frac{4}{5}\cdot\det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$= \boxed{-4} \quad (\text{using } \det I^{3} = 1).$$

<u>shortcut: "triangular" matrices</u> (to go straight from (*) to answer)

no need to go all the way to RREF if all you need is det A!

$$\det\begin{pmatrix}\lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n\end{pmatrix} = \lambda_1\lambda_2\cdots\lambda_n.$$

<u>eg</u> $\det\begin{pmatrix}3&7&\pi\\0&1&10^6\\0&0&2\end{pmatrix} = 3\cdot1\cdot2 = 6.$

reason: can cancel above pivots w/ Ri += c·Rj <u>only</u>,
& the scale $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ to I.

Monday.

<u>eg</u>2 $\det\begin{pmatrix}1&0&0&0\\1&1&1&1\\1&2&4&8\\1&3&9&27\end{pmatrix} \overset{(R2,R3,R4\ -=R1)}{=} \det\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&2_{-2}&4_{-2}&8_{-2}\\0&3_{-3}&9_{-3}&27_{-3}\end{pmatrix}$

$$\overset{\substack{R3-=2R2\\R4-=3R2}}{=} \det\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&2&6\\0&0&6_{-6}&24_{-18}\end{pmatrix}$$

$$\overset{R4-=3R3}{=} \det\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&2&6\\0&0&0&6\end{pmatrix} = 1\cdot1\cdot2\cdot6$$

$$= \boxed{12}$$

<u>eg3</u> rederivation of 2×2 formula: (when $a \neq 0$)

$$\det\begin{pmatrix} a & b \\ \underset{-c}{c} & \underset{-\frac{c}{a}\cdot b}{d} \end{pmatrix} \overset{R2 -= \frac{c}{a}R1}{=} \begin{pmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{pmatrix} = a\cdot\left(d-\frac{bc}{a}\right)$$

$$= \underline{ad-bc}.$$

Method 2 Cofactor expansion. (recursive... expresses n×n determinant w/ (n-1)×(n-1) determinants)

For n×n matrix A, the cofactor of entry (i,j) is

$$C_{ij} := (-1)^{i+j}\cdot\left[\text{det. of A w/ row } i \text{ \& column } j \text{ removed}\right].$$

↖ visualize this as: $\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

<u>Fact</u> for any row R_i, in an n×n matrix A,

$$\det A = \sum_{j=1}^{n} a_{ij}\cdot C_{ij}.$$ ("expanding along a row")

& for any column j,

$$\det A = \sum_{i=1}^{n} a_{ij}\cdot C_{ij}.$$

<u>eg1</u> (again) expanding along row 1:

$$\det\begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & 2 \\ 2 & 9 & 6 \end{pmatrix} = 0\cdot\det\begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix} - 5\cdot\det\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} + 2\cdot\det\begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}$$

$$= 0 - 5\cdot(1\cdot6-2\cdot2) + 2\cdot(1\cdot9-3\cdot2)$$ (2×2 formula)

$$= -5\cdot2+2\cdot3 = \boxed{-4}$$ (as we found before!)

M 9/30: begin here.

Notice: 0's are great! You can skip terms:

briefly discussed Friday: finish Monday.

eg. 4

$$\det\begin{pmatrix}5&\boxed{0}&0&1\\6&\boxed{2}&17&18\\7&\boxed{0}&1&19\\4&\boxed{0}&0&2\end{pmatrix}$$

expand along column 2:

mind the sign!

$$=-0\cdot\det\left(\text{don't care!}\right)+2\cdot\det\begin{pmatrix}5&\boxed{0}&1\\7&\boxed{1}&19\\4&\boxed{0}&2\end{pmatrix}-0\cdot\det\left(\text{don't care!}\right)+0\det(\text{"})$$

$$=2\cdot\det\begin{pmatrix}5&\boxed{0}&1\\7&\boxed{1}&19\\4&\boxed{0}&2\end{pmatrix}$$

$$=2\cdot\left[-0+1\cdot\det\begin{pmatrix}5&1\\4&2\end{pmatrix}-0\right]$$

$$=2\cdot(5\cdot2-1\cdot4)=\boxed{12}$$

qu which method is better?

ans It depends!

- Cofactors are great when a row or column has lots of 0's, and works quickly for 3x3 matrices.
- Row-reduction is better (usually) for bigger matrices, or when you see an operation that simplifies things.
- Sometimes, a hybrid approach is best. (do a row op, then expand).

R2 -= 2R1
R3 -= 3R1
R4 -= 4R1

eg. $$\det\begin{pmatrix} 1 & 7 & 2 & 5 \\ 2 & 16 & 4 & 10 \\ 3 & 4 & 11 & 15 \\ 4 & 40 & 78 & 27 \end{pmatrix} = \det\begin{pmatrix} 1 & 7 & 2 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & -17 & 5 & 0 \\ 0 & 12 & 70 & 7 \end{pmatrix}$$

(expand col. 1)

$$= 1\cdot\det\begin{pmatrix} 2 & 0 & 0 \\ -17 & 5 & 0 \\ 12 & 70 & 7 \end{pmatrix} - 0 + 0 - 0$$

$$= 1\cdot(2\cdot 5\cdot 7) = \boxed{70}$$