Amherst College
Department of Mathematics and Statistics

MATH 272 | MIDTERM 2 | FALL 2019

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back.
- No calculators or other devices are permitted.
- You may use any of the blank pages to continue answers if you run out of space. Please clearly indicate on the problem's original page if you do so, so that I know to look for it.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	Total
Points:	9	9	9	9	9	45
Score:						

1. [9 points] **Short answer questions.** No explanations are necessary.

(a) The set $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$. Compute the coordinates $\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right]_B$.

$$1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

so $\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right]_B = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}.}$

(b) Give an example of an inner product on the vector space $C[-\pi, \pi]$, by writing a formula for the inner product of two functions $f(x)$ and $g(x)$ below.

$$\boxed{\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx}$$

(c) Find the scalar multiple of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ that is closest to $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ (in the sense of minimizing the norm of the difference).

$$\operatorname{proj}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{10+0}{1+4} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}$$

2. [9 points] Consider the following two bases for $\mathbb{R}^2$ (you may assume that these are indeed bases).

$$\begin{aligned} B &= \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \\ B' &= \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\} \end{aligned}$$

Determine the transition matrix $[I]_B^{B'}$.

$$\left([I]_{B'}^{S}\right)^{-1}[I]_B^S$$

$$= \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= \frac{1}{(-2)} \begin{pmatrix} 7 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} 8 & 2 \\ -4 & -2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -4 & -1 \\ 2 & 1 \end{pmatrix}}$$

3. [9 points] Let W be the following subset of $\mathbb{R}^3$.

$$W = \left\{ \begin{pmatrix} r+s \\ r+t \\ s-t \end{pmatrix} : r,s,t \in \mathbb{R} \right\}$$

(a) Show that W is a subspace of $\mathbb{R}^3$ (in class I referred to sets like this as "parameterized subspaces").

W is nonempty, since e.g. $\begin{pmatrix}0\\0\\0\end{pmatrix} \in W$ (take $r=s=t=0$).

$\forall \vec{u}, \vec{v} \in W$. $\exists$ scalars $r,s,t,\ r',s',t'$ st. $\vec{u} = \begin{pmatrix} r+s \\ r+t \\ s-t \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} r'+s' \\ r'+t' \\ s'-t' \end{pmatrix}$,

hence $\forall c \in \mathbb{R}$ $\quad \vec{u} + c\vec{v} = \begin{pmatrix} r+s+c(r'+s') \\ r+t+c(r'+t') \\ s-t+c(s'-t') \end{pmatrix} = \begin{pmatrix} (r+cr') + (s+cs') \\ (r+cr') + (t+ct') \\ (s+cs') - (t+ct') \end{pmatrix}$,

which is in W (scalars $r+cr'$, $s+cs'$, $t+ct'$).

So W is a subspace of $\mathbb{R}^3$.

(b) Find a basis for W, and determine $\dim W$.

$$W = \left\{ r\begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\0\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\-1\end{pmatrix} : r,s,t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\}$$

shrink to basis:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

pivot columns 1 & 2

$\implies \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ is a basis for W,

& $\dim W = 2$.

4. [9 points] Suppose that $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set of vectors in a vector space V, and that $c_1, c_2, c_3, d_1, d_2, d_3$ are scalars such that

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = d_1\vec{u} + d_2\vec{v} + d_3\vec{w}.$$

Prove that $c_1 = d_1, c_2 = d_2$, and $c_3 = d_3$.

subtracting the RHS from the LHS,

$$(c_1-d_1)\vec{u} + (c_2-d_2)\vec{v} + (c_3-d_3)\vec{w} = \vec{0}.$$

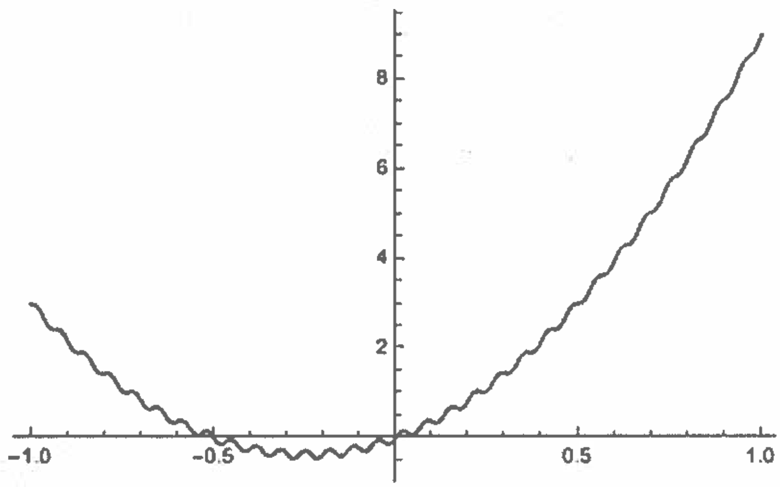
Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is LI, it follows that

$$c_1-d_1 = c_2-d_2 = c_3-d_3 = 0,$$

ie.

$c_1=d_1$, $c_2=d_2$ & $c_3=d_3$, as desired.

5. [9 points] A signal is measured in a lab, giving a function $f(x)$ on the interval $[-1,1]$ shown below. Based on the graph, you guess that this signal can be approximated by a linear combination of x, and x^2. In order to produce a good model, you numerically compute the following integrals.



$$\int_{-1}^{1} x \cdot f(x)\, dx = 2.00$$
$$\int_{-1}^{1} x^2 \cdot f(x)\, dx = 2.40$$

Regard $f(x), x$, and x^2 as elements of the inner product space $C[-1,1]$, with the usual inner product.

(a) Compute the projections $\text{proj}_x f(x)$ and $\text{proj}_{x^2} f(x)$.

$\langle x, f\rangle = 2$ (given) & $\langle x, x\rangle = \int_{-1}^{1} x^2 dx$
$= \left[\frac{1}{3}x^3\right]_{-1}^{1} = 2/3,$

so $\text{proj}_x f(x) = \frac{\langle x,f\rangle}{\langle x,x\rangle}\cdot x = \frac{2}{2/3}x = \boxed{3x}$

$\langle x^2, f\rangle = \cancel{3}\ 2.4$ (given) & $\langle x^2, x^2\rangle = \int_{-1}^{1} x^4 dx = \left[\frac{1}{5}x^5\right]_{-1}^{1} = 2/5$

so $\text{proj}_{x^2} f(x) = \frac{2.4}{2/5}x^2 = \frac{12}{2}x^2$
$= \boxed{6x^2}$

(b) Show that $x \perp x^2$ in this inner product space.

$$\langle x, x^2 \rangle = \int_{-1}^{1} x^3 dx = \left[\tfrac{1}{4}x^4\right]_{-1}^{1} = 0.$$

So $x \perp x^2$.

(c) Find coefficients a_1, a_2 minimizing the following objective function.

$$\int_{-1}^{1} \left(a_1 x + a_2 x^2 - f(x)\right)^2 \, dx. \quad = \|a_1x + a_2x^2 - f(x)\|^2$$

The function $g(x) = a_1 x + a_2 x^2$ will be a quadratic function approximating $f(x)$ as well as possible in the "least squares" sense.

Since $x \perp x^2$, the LC $\overset{g(x)}{}$ of $\{x, x^2\}$

minimizing $\|g(x) - f(x)\|^2$

is the sum of the projections:

$$g(x) = \text{proj}_x f + \text{proj}_{x^2} f$$

$$= 3x + 6x^2$$

ie. $\underline{a_1 = 3 \ \& \ a_2 = 6}$

Alternatively, the normal eq'n is

$$\begin{pmatrix} \langle x, x\rangle & \langle x, x^2\rangle \\ \langle x^2, x\rangle & \langle x^2, x^2\rangle \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \langle x, f\rangle \\ \langle x^2, f\rangle \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 2/3 & 0 \\ 0 & 2/5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.4 \end{pmatrix}$$

which gives the same sol'n.