1. [9 points] **Short answer questions.** No explanations are necessary.
   (a) State the dimension of each of the following vector spaces.
      - $\mathbb{R}^5$
      - $M_{2\times 3}$
      - $\mathcal{P}_3$

   (b) Find the angle between the vectors $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

   (c) For each of the following subsets of $\mathbb{R}^2$, determine whether or not it is a subspace of $\mathbb{R}^2$. You do not need to prove your answer; simply state "yes" or "no."
      - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \; x + 5y = 0 \right\}$
      - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \; y = x^2 \right\}$
      - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \; 2x + 3y = 1 \right\}$
      - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \; y = 7x \right\}$

2. [9 points] Consider the two bases $B = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

3. [15 points] Consider the following three vectors in $\mathbb{R}^4$.

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

   Denote by W the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.
   (a) Find a basis of W.
   (b) What is the dimension of W?
   (c) Find an *orthonormal* basis for W. (Recall that a basis is called *orthonormal* if any two vectors in the basis are orthogonal, and each vector has norm equal to 1.)
   (d) What element of W is closest to the vector $\vec{b} = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \end{pmatrix}$?

   (In other words, which element $\vec{x}$ of W minimizes $\|\vec{x} - \vec{b}\|$?)

4. [9 points] Suppose that $\{\vec{u}, \vec{v}\}$ is a basis for a vector space V. Prove that $\{3\vec{u} + 2\vec{v},\ \vec{u} + \vec{v}\}$ is also a basis for V.

5. [9 points] Let $V = \mathcal{P}_2$ (polynomials of degree at most 2), and equip V with an inner product defined as follows.

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)\ dx$$

   (a) Compute $\|x\|$ (the norm of $x \in \mathcal{P}_2$, according to this inner product).

   (b) Compute $\langle x, x^2 + 1 \rangle$.

   (c) Compute $\text{proj}_x(x^2 + 1)$.

   (d) The element $\text{proj}_x(x^2 + 1)$ solves a minimization problem. Carefully state this problem (what is being minimized)?

6. [9 points] Consider the following three vectors.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

   (a) Show that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set.

   (b) In fact, B is a basis (you don't need to prove this). Find the coordianates of the following vector in basis B.

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$