

1. [9 points] **Short answer questions.** No explanations are necessary.

(a) The set $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for \mathbb{R}^2 . Compute the coordinates $\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right]_B$.

(b) Give an example of an inner product on the vector space $\mathcal{C}[-\pi, \pi]$ of continuous functions on $[-\pi, \pi]$, by writing a formula for the inner product of two functions $f(x)$ and $g(x)$ below.

$$\langle f(x), g(x) \rangle =$$

(c) Find the scalar multiple of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ that is closest to $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ (in the sense of minimizing the norm of the difference).

2. [9 points] Consider the following two bases for \mathbb{R}^2 (you may assume that these are indeed bases).

$$B = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$$

$$B' = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$$

Determine the transition matrix $[I]_B^{B'}$.

3. [9 points] Let W be the following subset of \mathbb{R}^3 .

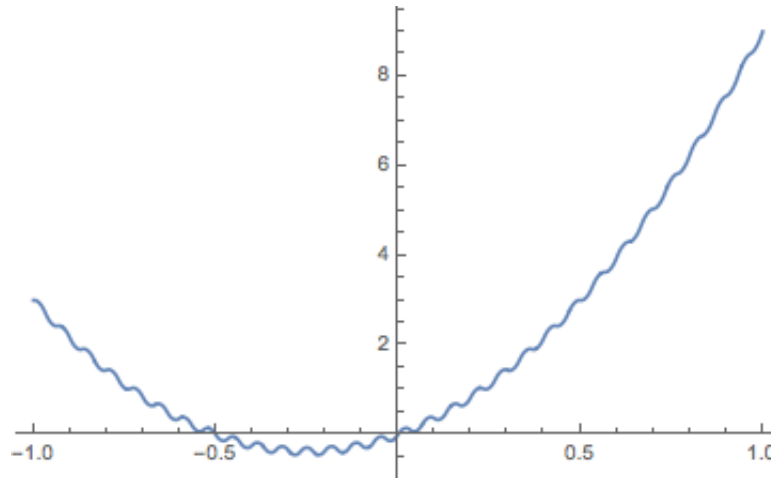
$$W = \left\{ \begin{pmatrix} r+s \\ r+t \\ s-t \end{pmatrix} : r, s, t \in \mathbb{R} \right\}$$

- (a) Show that W is a subspace of \mathbb{R}^3 (in class I referred to sets like this as “parameterized subspaces”).
- (b) Find a basis for W , and determine $\dim W$.
4. [9 points] Suppose that $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set of vectors in a vector space V , and that $c_1, c_2, c_3, d_1, d_2, d_3$ are scalars such that

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = d_1\vec{u} + d_2\vec{v} + d_3\vec{w}.$$

Prove that $c_1 = d_1$, $c_2 = d_2$, and $c_3 = d_3$.

5. [9 points] A signal is measured in a lab, giving a function $f(x)$ on the interval $[-1, 1]$ shown below. Based on the graph, you guess that this signal can be approximated by a linear combination of x and x^2 .



In order to produce a good model, you numerically compute the following integrals.

$$\int_{-1}^1 x \cdot f(x) \, dx = 2.00$$

$$\int_{-1}^1 x^2 \cdot f(x) \, dx = 2.40$$

Regard $f(x)$, x , and x^2 as elements of the inner product space $\mathcal{C}[-1, 1]$, with the usual inner product.

- Compute the projections $\text{proj}_x f(x)$ and $\text{proj}_{x^2} f(x)$.
- Show that $x \perp x^2$ in this inner product space.
- Find coefficients a_1, a_2 minimizing the following objective function.

$$\int_{-1}^1 (a_1x + a_2x^2 - f(x))^2 \, dx.$$

The function $g(x) = a_1x + a_2x^2$ will be a quadratic function approximating $f(x)$ as well as possible in the “least squares” sense.