Amherst College
Department of Mathematics and Statistics

MATH 272 | FINAL EXAM | SPRING 2019

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back.
- No calculators or other devices are permitted.
- You may use any of the blank pages to continue answers if you run out of space. Please clearly indicate on the problem's original page if you do so, so that I know to look for it.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	9	9	10	9	10	10	9	14	90
Score:										

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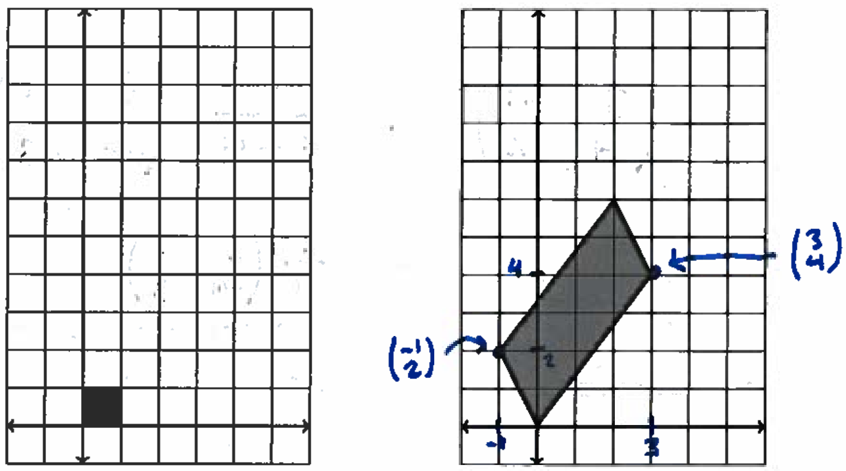
1. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that has the following effect on the unit square.



(a) [2 points] Find the matrix representation of T in the standard basis.
(There are two possible answers based on how you interpret the picture. You need only give one.)

$T(\vec{e}_1)$ & $T(\vec{e}_2)$ are $\binom{3}{4}$ & $\binom{-1}{2}$ in some order.

So $[T]_S = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$ or $\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$.

(either answer suffices).

(b) [2 points] Determine $R(T)$ and $N(T)$ (no explanation is necessary for this part).

$R(T) = \mathbb{R}^2$

& $N(T) = \{\vec{0}\}$

since T is invertible.

(continued on reverse)

(c) [3 points] Find the matrix representation, in the standard basis, of the inverse transformation T^{-1}.

$$\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}^{-1} = \frac{1}{6+4} \cdot \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

OR (other ans. in (a))

$$\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{-4-6} \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} -4 & 3 \\ 2 & 1 \end{pmatrix}$$

(d) [3 points] Determine a point $\vec{x} \in \mathbb{R}^2$ that is sent to $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ by this transformation (i.e. $T(\vec{x}) = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$).

$$T^{-1}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 10+10 \\ -20+30 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

OR (w/ other answer in (a))

$$T^{-1}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

2. Let A be the following 3×5 matrix.

$$A = \begin{pmatrix} 2 & -6 & -9 & -11 & -8 \\ 2 & -6 & -6 & -8 & -4 \\ 1 & -3 & -3 & -4 & 0 \end{pmatrix}$$

The reduced row echelon form of A is as follows (you do not need to verify this yourself).

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) [3 points] Give a basis for $N(A)$.

$\vec{x} \in N(A) \iff x_1 = 3x_2 + x_4$, $x_3 = -x_4$, $x_5 = 0$ (x_2, x_4 free)

$\iff \vec{x} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

$\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $N(A)$.

(b) [3 points] Give a basis for $R(A)$.

Pivots in cols. 1,3,5 of RREF

$\Rightarrow$ cols. 1,3,5 of original span $R(A)$

$\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -9 \\ -6 \\ -3 \end{pmatrix}, \begin{pmatrix} -8 \\ -4 \\ 0 \end{pmatrix} \right\}$ is a basis for $R(A)$.

(continued on reverse)

(c) [3 points] Let A be the same matrix above, and let $\vec{b}$ be any vector in $\mathbb{R}^3$. Explain why the matrix equation $A\vec{x} = \vec{b}$ is consistent, regardless of the choice of $\vec{b}$. How many free variables occur in the general solution of $A\vec{x} = \vec{b}$?

Since $\dim R(A) = 3 = \dim \mathbb{R}^3$, in fact

$R(A) = \mathbb{R}^3$.

Hence $\forall \vec{b} \in \mathbb{R}^3$, $\vec{b} \in R(A)$,

ie. $\exists \vec{x} \in \mathbb{R}^4$ st. $A\vec{x} = \vec{b}$

ie. $A\vec{x} = \vec{b}$ is consistent.

Any two sol'ns differ by an el't of $N(A)$,

ie. a LC of the basis vectors in part (a).

There are two such vectors. hence two free variables

in the gen'l sol'n to $A\vec{x} = \vec{b}$ (regardless of $\vec{b}$).

3. [9 points] Let

$$A = \begin{pmatrix} 2 & -2 & -3 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}.$$

Determine the inverse matrix A^{-1}.

Solving $AX = I$ by row-reduction:

$$\left(\begin{array}{ccc|ccc} 2_{-2} & -2_{-2} & -3_{-4} & 1 & 0_{-2} & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 3_{-3} & 0_{-3} & 1_{-6} & 0 & 0_{-3} & 1 \end{array}\right)$$

$R1 -= 2R2$
$R3 -= 3R2$
$R1 \leftrightarrow R2$
$\downarrow$

$$\left(\begin{array}{ccc|ccc} 1 & 1_{-1} & 2_{-7/4} & 0_{+1/4} & 1_{-1/2} & 0 \\ 0 & -4 & -7 & 1 & -2 & 0 \\ 0 & -3_{+3} & -5_{+3\cdot 7/4} & 0_{-3/4} & -3_{+3/2} & 1 \end{array}\right)$$

$R1 += \frac{1}{4}R2$
$R3 -= \frac{3}{4}R2$
$R2 *= (-\frac{1}{4})$
$\downarrow$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/4_{-1/4} & 1/4_{+3/4} & 1/2_{+3/2} & 0_{-1} \\ 0 & 1 & 7/4_{-7/4} & -1/4_{+21/4} & +1/2_{+21/2} & 0_{-7} \\ 0 & 0 & 1/4 & -3/4 & -3/2 & 1 \end{array}\right)$$

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$R1 -= R3$
$R2 -= 7R3$
$R3 *= 4$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 5 & 11 & -7 \\ 0 & 0 & 1 & -3 & -6 & 4 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 5 & 11 & -7 \\ -3 & -6 & 4 \end{pmatrix}$$

4. Suppose that $T : V \to W$ is a linear transformation of vector spaces, and $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a set of three vectors in V.

(a) [5 points] Suppose $\{T(\vec{v}_1),\ T(\vec{v}_2),\ T(\vec{v}_3)\}$ is a linearly independent set in W. Prove that S is a linearly independent set in V.

We must show that no nontrivial LC of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is $\vec{0}$.

Suppose that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}. \quad \text{(we'll show } c_1 = c_2 = c_3 = 0\text{)}.$$

Then since T is linear, it passes through LC's & sends $\vec{0}$ to $\vec{0}$:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0})$$

$$\Rightarrow \quad c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = \vec{0}.$$

Since $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is LI, this LC must be trivial, ie.

$$c_1 = c_2 = c_3 = 0,$$

as desired.

(continued on reverse)

(b) [5 points] Prove that if S is a linearly independent set in V and T is one-to-one, then $\{T(\vec{v}_1),\ T(\vec{v}_2),\ T(\vec{v}_3)\}$ is a linearly independent set in W.

Suppose that

$$c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + c_3 T(\vec{v}_3) = \vec{0}.$$ (we'll show this LC is trivial, i.e. $c_1 = c_2 = c_3 = 0$).

As in (a), we may rewrite this as

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0}).$$

Since T is one-to-one, equal outputs imply equal inputs & hence

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Now since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LI, this LC is trivial, i.e.

$$c_1 = c_2 = c_3 = 0,$$

as desired.

5. [9 points] Denote by $\vec{u}, \vec{v}, \vec{b}$ the following three vectors in $\mathbb{R}^4$.

$$\vec{u} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ -11 \\ -11 \\ 11 \end{pmatrix}$$

Determine the linear combination $\vec{w}$ of $\{\vec{u}, \vec{v}\}$ that is closest to $\vec{b}$ (that is, the linear combination that minimizes $\|\vec{w} - \vec{b}\|$).

By the normal eq'n,

$a\vec{u} + b\vec{v}$ is closest to $\vec{b}$

$$\iff \begin{cases} \vec{u} \perp (a\vec{u} + b\vec{v} - \vec{b}) & \& \\ \vec{v} \perp (a\vec{u} + b\vec{v} - \vec{b}) \end{cases}$$

$$\iff \begin{pmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v} \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \vec{u}\cdot\vec{b} \\ \vec{v}\cdot\vec{b} \end{pmatrix}$$

$$\iff \begin{pmatrix} 1+4 & -1-2 \\ -1-2 & 1+1+1+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11-22 \\ -11+11-11 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -11 \\ -11 \end{pmatrix}$$

$$\iff \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} -11 \\ -11 \end{pmatrix} = \frac{1}{20-9} \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -11 \\ -11 \end{pmatrix}$$

$$= \frac{1}{11} \cdot \begin{pmatrix} -77 \\ -88 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix}$$

so the nearest LC is $\underline{-7\vec{u} - 8\vec{v}} = \begin{pmatrix} 0 \\ 7 \\ -14 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ 8 \\ -8 \\ -8 \end{pmatrix} = \underline{\begin{pmatrix} -8 \\ -1 \\ -6 \\ 8 \end{pmatrix}}$.

// can check by computing $\begin{pmatrix} -8 \\ -1 \\ -6 \\ 8 \end{pmatrix} - \vec{b} = \begin{pmatrix} -8 \\ 10 \\ 5 \\ -3 \end{pmatrix}$ & checking that this is $\perp \vec{u}$ & $\perp \vec{v}$.

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6. Let V be an inner product space. Suppose that W is a subspace of V with basis $B = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

(a) [2 points] What is the dimension of W?

$\dim W = 3$ (a basis B has 3 elements).

(b) [4 points] Suppose that $\vec{b}$ is a vector in V, and $\vec{u}$ is a vector in W such that

$$\vec{w}_1 \perp (\vec{b} - \vec{u}), \qquad \vec{w}_2 \perp (\vec{b} - \vec{u}), \qquad \text{and} \qquad \vec{w}_3 \perp (\vec{b} - \vec{u}).$$

Prove that $\vec{b} - \vec{u}$ is orthogonal to every vector in W.

$\forall \vec{w} \in W$, $\exists c_1, c_2, c_3 \in \mathbb{R}$ s.t. $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3$ (since B spans W)

$\Rightarrow \vec{w} \cdot (b - \vec{u}) = c_1(\vec{w}_1 \cdot (b - \vec{u})) + c_2(\vec{w}_2 \cdot (\vec{b} - \vec{u})) + c_3(\vec{w}_3 \cdot (\vec{b} - \vec{u}))$ (since dot product is linear)

$= c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0$ (given $\perp$ conditions)

$= 0$.

$\Rightarrow \vec{w} \perp (\vec{b} - \vec{u})$,

as desired.

(continued on reverse)

(c) [4 points] Let $\vec{b}, \vec{u}$ be as in part (b). Prove that if $\vec{v}$ is any other vector in W, then

$$\|\vec{b} - \vec{v}\|^2 = \|\vec{b} - \vec{u}\|^2 + \|\vec{u} - \vec{v}\|^2.$$

Observe that $\vec{u}, \vec{v} \in W \Rightarrow \underline{\vec{u} - \vec{v} \in W}$

(since subspaces are closed under subtraction)

hence by part (b), $\underline{(\vec{u}-\vec{v}) \perp (\vec{b}-\vec{u})}$.

Thus by the Pythag. thm. for inner product spaces,

$$\| (\not{\vec{u}}-\vec{v}) + (b-\not{\vec{u}}) \|^2 = \|u-v\|^2 + \|b-\vec{u}\|^2$$

ie.

$$\| \vec{b}-\vec{v} \|^2 = \|\vec{u}-\vec{v}\|^2 + \| \vec{b}-\vec{u}\|^2,$$

as desired.

7. Define a map $T : \mathcal{P}_2 \to \mathbb{R}^4$ by the formula

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{pmatrix}.$$

T is a linear transformation (you do not need to prove this).

(a) [4 points] Let $B = \{(x-1)(x-2),\ x(x-2),\ x(x-1)\}$. This is a basis of $\mathcal{P}_2$ (you do not need to prove this). Let S denote the standard basis of $\mathbb{R}^4$. Determine the matrix representation $[T]_B^S$ of T with respect to the basis B of $\mathcal{P}_2$ and the standard basis S of $\mathbb{R}^4$.

$$[T]_B^S = \begin{pmatrix} | & | & | \\ [T(b_1)]_S & [T(b_2)]_S & [T(b_3)]_S \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ T((x-1)(x-2)) & T(x(x-2)) & T(x(x-1)) \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} (0-1)(0-2) & 0\cdot(0-2) & 0\cdot(0-1) \\ (1-1)(1-2) & 1\cdot(1-2) & 1\cdot(1-1) \\ (2-1)(2-2) & 2\cdot(2-2) & 2\cdot(2-1) \\ (3-1)(3-2) & 3\cdot(3-2) & 3\cdot(3-1) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 2 & 3 & 6 \end{pmatrix}}$$

(b) [3 points] Let A be the matrix $[T]_B^S$ obtained in the previous part. Prove that $\vec{b} \in R(T)$ if and only if the matrix equation $A\vec{x} = \vec{b}$ is consistent.

$\vec{b} \in R(T) \iff \exists\, p(x) \in \mathcal{P}_2$ st $T(p(x)) = \vec{b}$ (def'n of $R(T)$)

$\iff \exists\, p(x) \in \mathcal{P}_2$ st. $[T(p(x))]_S = \vec{b}$ (since $[\vec{b}]_S = \vec{b}$)

$\iff \exists\, p(x) \in \mathcal{P}_2$ st. $[T]_B^S [p(x)]_B = \vec{b}$ (def'n of $[T]_B^S$)

$\iff \exists\, \vec{x} \in \mathbb{R}^3$ st. $[T]_B^S \vec{x} = \vec{b}$ (taking coords is invertible)

$\iff \exists\, \vec{x} \in \mathbb{R}^3$ st. $A\vec{x} = \vec{b}$

$\iff A\vec{x} = \vec{b}$ is consistent.

(c) [3 points] Suppose that we are given four constants a, b, c, d, and wish to find a polynomial $p(x) \in \mathcal{P}_2$ whose graph passes through the four points $(0, a), (1, b), (2, c), (3, d)$. For which values of a, b, c, d is this possible? Express your answer as a linear equation in a, b, c, and d.

Hint: interpret this as asking when a linear system is consistent, using the previous part.

$y = p(x)$ passes through these pts. iff $\begin{pmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$\iff T(p(x)) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \iff A\vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is consistent (part (b)).

Row-reducing gives:

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & 2 & c \\ 2 & 3 & 6 & d \end{array}\right) \xrightarrow[\substack{R4 -= R1 \\ R4 += 3R2 \\ R4 -= 3R3}]{} \left(\begin{array}{ccc|c} 2 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & 2 & c \\ 0 & 0 & 0 & d-a+3b-3c \end{array}\right)$$

(under the 4th row: $-2 \quad -3 \quad -6 \quad -a+3b-3c$)

This is in REF (though not RREF), and is consistent iff the last row doesn't read $0=1$, ie.

such $p(x)$ exists iff $d - a + 3b - 3c = 0$

ie. $\boxed{-a + 3b - 3c + d = 0}$

8. [9 points] Let A be an $n \times n$ matrix, and λ a scalar. Define as usual the following set (called the *eigenspace* in class).

$$V_\lambda = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

Prove that V_λ is a *subspace* of $\mathbb{R}^n$.

By a result in class, V_λ is a subspace iff it is nonempty, closed under addition, & closed under scalar multiplication.

We will check each of these three criteria.

nonemptiness $A \cdot \vec{0} = \vec{0} = \lambda \cdot \vec{0}$, so $\vec{0} \in V_\lambda$, hence V_λ is nonempty.

closure under + Suppose $\vec{v}, \vec{w} \in V_\lambda$, i.e. $A\vec{v} = \lambda\vec{v}$ & $A\vec{w} = \lambda\vec{w}$.

Then $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$ (matrix mult. is distributive)

$= \lambda\vec{v} + \lambda\vec{w}$ (since $\vec{v}, \vec{w} \in V_\lambda$)

$= \lambda(\vec{v}+\vec{w})$ (scalar mult. is distributive)

so $\vec{v}+\vec{w} \in V_\lambda$.

closure under scalar · Suppose $\vec{v} \in V_\lambda$ and $c \in \mathbb{R}$.

Then $A(c\vec{v}) = c(A\vec{v})$ (scalar mult. commutes w/ matrix mult.)

$= c(\lambda\vec{v})$ ($\vec{v} \in V_\lambda$)

$= (c\lambda)\vec{v} = \lambda(c\vec{v})$ (scalar mult. is associative)

so $c\vec{v} \in V_\lambda$.

So the three criteria hold, & V_λ is indeed a subspace.

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9. In a certain town, 8000 customers subscribe to internet service from company A. A competitor, company B, enters the market and begins to draw customers from company A.

Suppose that every year, 10% of company A's customers change their service to company B, and 30% of company B's customers change their service to company A.

For example:

- In the first year, 800 customers (10% of 8000) change service from A and B, after which company A has 7200 customers and company B has 800 customers.
- In the second year, 720 customers (10% of 7200) switch from A to B, and 240 customers (30% of 800) switch from B to A. So after two years, company A has $7200 - 720 + 240 = 6720$ customers and company B has $800 + 720 - 240 = 1280$ customers.

(a) [2 points] Find a 2×2 matrix M encoding the change in number of customers of each company from one year to the next. More precisely: if a, b denote the number of customers of companies A and B (respectively) in a given year, and a', b' denote the number of customers in the following year, the matrix M should satisfy

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}.$$

You can check your answer by verifying that

$$\begin{pmatrix} 7200 \\ 800 \end{pmatrix} = M \begin{pmatrix} 8000 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6720 \\ 1280 \end{pmatrix} = M^2 \begin{pmatrix} 8000 \\ 0 \end{pmatrix}.$$

$$\begin{cases} a' = (1-0.1)a + 0.3b \\ \& \; b' = 0.1a + (1-0.3)b \end{cases}$$

ie. $$\begin{cases} a' = 0.9a + 0.3b \\ \& \; b' = 0.1a + 0.7b \end{cases}$$

ie. $$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

so $$\boxed{M = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}}.$$

(continued on reverse)

(b) [6 points] Find the eigenvalues of M, and an eigenvector for each eigenvalue.

char. eq'n:

$$\det(M - \lambda I) = 0$$

$$\iff \det\begin{pmatrix} 0.9-\lambda & 0.3 \\ 0.1 & 0.7-\lambda \end{pmatrix} = 0$$

$$\iff (0.9-\lambda)(0.7-\lambda) - 0.3 \cdot 0.1 = 0$$

$$\iff \lambda^2 - 1.6\lambda + 0.63 - 0.03 = 0$$

$$\iff \lambda^2 - 1.6\lambda + 0.6 = 0$$

$$\iff (\lambda - 1)(\lambda - 0.6) = 0$$

so there are two eigenvalues: $\lambda = 1$ & $\lambda = 0.6$.

$\underline{\lambda = 1}$

$$V_1 = N\begin{pmatrix} 0.9-1 & 0.3 \\ 0.1 & 0.7-1 \end{pmatrix} = N\begin{pmatrix} -0.1 & 0.3 \\ 0.1 & -0.3 \end{pmatrix}$$

$$= N\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$ (row op $R2 += R1$, $R1 *= (-\frac{1}{0.1})$)

$$= \text{span}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\lambda = 1$ has eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\underline{\lambda = 0.6}$

$$V_{0.6} = N\begin{pmatrix} 0.9-0.6 & 0.3 \\ 0.1 & 0.7-0.6 \end{pmatrix} = N\begin{pmatrix} 0.3 & 0.3 \\ 0.1 & 0.1 \end{pmatrix}$$

$$= N\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$ (row op $R2 -= \frac{1}{3}R1$, $R1 *= (\frac{1}{0.3})$)

$$= \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\lambda = 0.6$ has eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(c) [3 points] Express $\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$ as a linear combination of the eigenvectors you found in part (b).

$$\left(\begin{array}{cc|c} 3 & -1 & 8000 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R2 -= \frac{1}{3}R1} \left(\begin{array}{cc|c} 3 & -1 & 8000 \\ 0 & 4/3 & 8000/3 \end{array}\right)$$

$-1 \qquad +1/3 \qquad -8000/3$

$$\xrightarrow[\substack{R2 *= 3/4 \\ R1 += R2 \\ R1 *= 1/3}]{} \left(\begin{array}{cc|c} 1 & 0 & 2000 \\ 0 & 1 & -2000 \end{array}\right)$$

(or compute $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$)

so $\boxed{\begin{pmatrix} 8000 \\ 0 \end{pmatrix} = 2000 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2000\begin{pmatrix} -1 \\ 1 \end{pmatrix}.}$

(d) [3 points] Find a formula for $M^n\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$, and use your formula to evaluate $\lim_{n\to\infty} M^n\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$.

$$M^n\begin{pmatrix} 8000 \\ 0 \end{pmatrix} = 2000\ M^n\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2000 \cdot M^n\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(linearity)

$$= 2000 \cdot 1^n\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2000 \cdot (0.6)^n\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= 2000\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2000 \cdot (0.6)^n\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6000 \\ 2000 \end{pmatrix} + (0.6)^n\begin{pmatrix} +2000 \\ -2000 \end{pmatrix}$$

useful ~~place~~ way to check answer: $n=0$ gives $\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$, $n=1$ gives $\begin{pmatrix} 7200 \\ 800 \end{pmatrix}$, $n=2$ gives $\begin{pmatrix} 6720 \\ 1280 \end{pmatrix}$.

so the steady-state is

$$\lim_{n\to\infty} M^n\begin{pmatrix} 8000 \\ 0 \end{pmatrix} = \begin{pmatrix} 6000 \\ 2000 \end{pmatrix} + \left[\lim_{n\to\infty}(0.6)^n\right] \cdot \begin{pmatrix} 2000 \\ -2000 \end{pmatrix}$$

$$= \begin{pmatrix} 6000 \\ 2000 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2000 \\ -2000 \end{pmatrix} = \boxed{\begin{pmatrix} 6000 \\ 2000 \end{pmatrix}}$$

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