Amherst College
Department of Mathematics and Statistics

MATH 272 FINAL EXAM SPRING 2019

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back.
- No calculators or other devices are permitted.
- You may use any of the blank pages to continue answers if you run out of space. Please clearly indicate on the problem's original page if you do so, so that I know to look for it.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	9	9	9	9	9	9	9	9	9	81
Score:										

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1. [9 points] Evaluate the **determinant** of the following matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 1-1 & 3-2 & 1 & 3-2 \\ 0 & 1 & 2 & 4 \end{pmatrix} \overset{R3-=R1}{=} \det\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1_{-1} & 1_{+1} & 1_{-1} \\ 0 & 1_{-1} & 2_{+1} & 4_{-1} \end{pmatrix}$$

$$\overset{\substack{R3-=R2 \\ R4-=R2}}{=} \det\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3_{-3} & 3 \end{pmatrix}$$

$$\overset{R4-=\frac{3}{2}R3}{=} \det\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$= 1\cdot 1\cdot 2\cdot 3$ (det. of triangular matrix is product of diagonal elements).

$= \boxed{6}$

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2. [9 points] Define a linear operator $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ by the following formula.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 9x - 4y \\ 25x - 11y \end{bmatrix}$$

(a) Let S denote the standard basis of $\mathbb{R}^2$. Determine the matrix representation $[T]_S$.

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 9 & -4 \\ 25 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow [T]_S = \boxed{\begin{pmatrix} 9 & -4 \\ 25 & -11 \end{pmatrix}}$$

(b) Let $B = \left\{\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$. This is a basis for $\mathbb{R}^2$ (you do not need to prove this). Determine the matrix representation $[T]_B$.

1^{st} column: $\left[T\left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right)\right]_B = \left[\begin{pmatrix} 18-20 \\ 50-55 \end{pmatrix}\right]_B = \left[\begin{pmatrix} -2 \\ -5 \end{pmatrix}\right]_B$

$= \underline{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}$ (since $\begin{pmatrix} -2 \\ -5 \end{pmatrix} = (-1)\cdot\begin{pmatrix} 2 \\ 5 \end{pmatrix} + 0\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix}$).

2^{nd} column: $\left[T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\right]_B = \left[\begin{pmatrix} 9-8 \\ 25-22 \end{pmatrix}\right]_B = \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]_B$

To find these coordinates: $\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 5 & 2 & 3 \end{array}\right) \xrightarrow{R2 -= \frac{5}{2}R1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -1/2 & 1/2 \end{array}\right)$

$\xrightarrow[R1 /= 2]{R1 += 2R2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right)$

$\Rightarrow \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]_B = \underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$

Hence $\boxed{[T]_B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}}$

<u>Alternate solution</u>: $[I]_B^S = \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$, so

$$[T]_B = [I]_S^B \cdot [T]_S \cdot [I]_B^S$$
$$= \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 9 & -4 \\ 25 & -11 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$$
$$= \frac{1}{(-1)}\begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 9 & -4 \\ 25 & -11 \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

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3. [9 points] Define three vectors in $\mathbb{R}^4$ as follows.

$$\vec{u} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad \vec{v} = \begin{pmatrix}0\\1\\1\\2\end{pmatrix} \qquad \vec{b} = \begin{pmatrix}5\\5\\6\\0\end{pmatrix}$$

Find the linear combination of $\vec{u}$ and $\vec{v}$ that is as close as possible to $\vec{b}$.

Normal eq'n:

The closest is $c_1\vec{u} + c_2\vec{v}$, where

$$\begin{pmatrix}\vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v}\\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\vec{u}\cdot\vec{b}\\ \vec{v}\cdot\vec{b}\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}1+0+1+0 & 0+0+1+0\\ 0+0+1+0 & 0+1+1+4\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}5+0+6+0\\ 0+5+6+0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}2 & 1\\ 1 & 6\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}11\\11\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 6\end{pmatrix}^{-1}\cdot\begin{pmatrix}11\\11\end{pmatrix}$$

$$= \frac{1}{2\cdot 6 - 1\cdot 1}\cdot\begin{pmatrix}6 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}11\\11\end{pmatrix}$$

$$= \frac{1}{11}\cdot\begin{pmatrix}55\\11\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}$$

So the closest LC is $5\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\\2\end{pmatrix} = \boxed{\begin{pmatrix}5\\1\\6\\2\end{pmatrix}}$

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4. [9 points] Define a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(\vec{v}) = A\vec{v}$, where A is the following matrix.

$$A = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & -2 & 0 & 5 \\ 1 & -2 & 0 & 5 \\ 1 & -2 & 1 & 8 \end{pmatrix}$$

(a) Find a basis for the **range** $R(T)$. What is its dimension?

row-reducing A gives: $\xrightarrow[-=R1]{R2,R3,R4} \begin{pmatrix} 1 & -2 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{pmatrix}$

$\xrightarrow[\substack{R3 -= R2 \\ R4 -= 2R2}]{R1 += R2} \begin{pmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Pivots in columns 1 & 3, so the 1st & 3rd columns of A are a basis for the range.

$$\boxed{\text{basis } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} ; \quad \dim R(T) = 2}$$

(b) Find a basis for the **null space** $N(T)$. What is its dimension?

Using the same RREF, the gen'l sol'n to $A\vec{x} = \vec{0}$ is

$x_1 = 2x_2 - 5x_4$

x_2 free

$x_3 = -3x_4$

x_4 free

ie.

$$\vec{x} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

$$\boxed{\text{basis } \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\} ; \quad \dim N(T) = 2}$$

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5. [9 points] Let A be the following matrix.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -1 & 3 \end{pmatrix}$$

(a) Determine the eigenvalues of A.

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & -2 \\ 0 & -1 & 3-\lambda \end{pmatrix} = (2-\lambda)\cdot\det\begin{pmatrix} 2-\lambda & -2 \\ -1 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)\cdot\left[(2-\lambda)(3-\lambda) - 2\right] = (2-\lambda)(\lambda^2 - 5\lambda + 6 - 2)$$

$$= (2-\lambda)(\lambda-4)(\lambda-1)$$

$\boxed{\lambda = 1, 2, 4}$ (three eigenvalues).

(b) Find a corresponding eigenvector for each eigenvalue.

$\lambda = 1$: $V_1 = N(A-I) = N\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} = N\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$= \text{span of } \boxed{\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}}$

$\lambda = 2$: $V_2 = N(A-2I) = N\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix} = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$= \text{span of } \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\lambda = 4$ $V_4 = N(A-4I) = N\begin{pmatrix} -2 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -1 & -1 \end{pmatrix} = N\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$= \text{span of } \boxed{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$

(continued on reverse)

Additional space for part (b), if needed.

(c) Diagonalize the matrix A. That is: determine a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

$$\boxed{P = \begin{pmatrix} -3 & 1 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$$

(change of basis $[I]_B^S$, where $B = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$, an eigenbasis).

(matrix rep $[A]_B$ in the eigenbasis).

(To check that this works, one can verify that
$$AP = PD = \begin{pmatrix} -3 & 2 & 0 \\ 2 & 0 & -4 \\ 1 & 0 & 4 \end{pmatrix}$$
).

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6. [9 points] Let V be an inner product space, and let $\vec{u}, \vec{b} \in V$ be two specific vectors in V.

(a) Recall the following definition: the projection of $\vec{b}$ onto $\vec{u}$ is

$$\text{proj}_{\vec{u}}\vec{b} = \frac{\langle \vec{u}, \vec{b} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}.$$

Prove, using this defintion, that $(\vec{b} - \text{proj}_{\vec{u}}\vec{b}) \perp \vec{u}$.

This is equivant to saying that $\langle \vec{b} - \text{proj}_{\vec{u}} \vec{b}, \vec{u} \rangle = 0$.

Using properties of inner products:

$$\langle \vec{b} - \text{proj}_{\vec{u}}\vec{b}, \vec{u} \rangle = \langle \vec{b}, \vec{u} \rangle - \langle \text{proj}_{\vec{u}}\vec{b}, \vec{u} \rangle \quad \text{(bilinearity)}$$

$$= \langle \vec{b}, \vec{u} \rangle - \left\langle \frac{\langle \vec{u}, \vec{b} \rangle}{\langle \vec{u}, \vec{u} \rangle} \cdot \vec{u}, \vec{u} \right\rangle \quad \text{(defn of proj.)}$$

$$= \langle \vec{b}, \vec{u} \rangle - \frac{\langle \vec{u}, \vec{b} \rangle}{\langle \vec{u}, \vec{u} \rangle} \cdot \langle \vec{u}, \vec{u} \rangle \quad \text{(bilinearity again)}$$

$$= \langle \vec{b}, \vec{u} \rangle - \langle \vec{u}, \vec{b} \rangle = \langle \vec{b}, \vec{u} \rangle - \langle \vec{b}, \vec{u} \rangle \quad \text{(symmetry)}$$

$$= 0, \text{ as desired.}$$

(b) Use part (a) to prove that for any constant c,

$$\|\vec{b} - c\vec{u}\|^2 = \|\vec{b} - \text{proj}_{\vec{u}}\vec{b}\|^2 + \|c\vec{u} - \text{proj}_{\vec{u}}\vec{b}\|^2$$

Observe that $\langle c\vec{u} - \text{proj}_{\vec{u}}\vec{b}, \vec{b} - \text{proj}_{\vec{u}}\vec{b} \rangle = \left(c - \frac{\langle \vec{u}, \vec{b} \rangle}{\langle \vec{u}, \vec{u} \rangle}\right) \cdot \langle \vec{u}, \vec{b} - \text{proj}_{\vec{u}}\vec{b} \rangle$

$= 0$, (by (a))

so $(\vec{b} - \text{proj}_{\vec{u}}\vec{b}) \perp (c\vec{u} - \text{proj}_{\vec{u}}\vec{b})$.

By the Pythag. thm. for inner product spaces,

$$\|(b - \text{proj}_{\vec{u}}\vec{b}) \mp (c\vec{u} - \text{proj}_{\vec{u}}\vec{b})\|^2 = \|\vec{b} - \text{proj}_{\vec{u}}\vec{b}\|^2 + \|c\vec{u} - \text{proj}_{\vec{u}}\vec{b}\|^2,$$

as desired.

(continued on reverse)

(c) Use part (b) to prove that the projection $\text{proj}_{\vec{u}}\vec{b}$ is closer to $\vec{b}$ than any other multiple of $\vec{u}$.

If $c\vec{u}$ is any other multiple of $\vec{u}$ (besides $\text{proj}_{\vec{u}}\vec{b}$), then

$$c\vec{u} \neq \text{proj}_{\vec{u}}\vec{b} \implies c\vec{u} - \text{proj}_{\vec{u}}\vec{b} \neq \vec{0}.$$

$$\implies \|c\vec{u} - \text{proj}_{\vec{u}}\vec{b}\| > 0$$ (positive definiteness of the inner product),

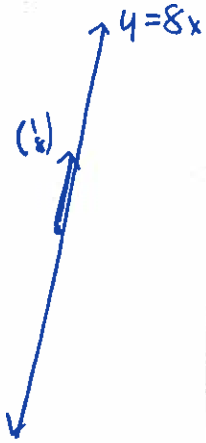
so using (b),

$$\|c\vec{u} - \vec{b}\| = \sqrt{\|\vec{b} - \text{proj}_{\vec{u}}\vec{b}\|^2 + \|c\vec{u} - \text{proj}_{\vec{u}}\vec{b}\|^2}$$
$$> \sqrt{\|\vec{b} - \text{proj}_{\vec{u}}\vec{b}\|^2} = \|\vec{b} - \text{proj}_{\vec{u}}\vec{b}\|,$$

so the distance between $c\vec{u}$ & $\vec{b}$ (as measured by the inner product) is strictly greater than the difference between $\text{proj}_{\vec{u}}\vec{b}$ & $\vec{b}$, as desired.

7. [9 points] Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line $y = 8x$.

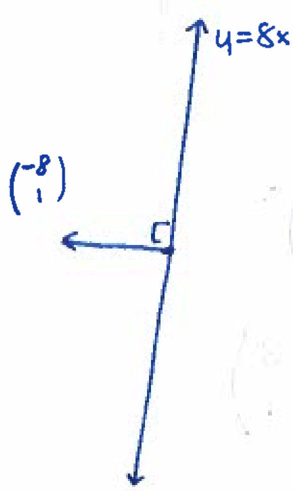
(a) Find a nonzero vector $\vec{u}$ such that $T(\vec{u}) = \vec{u}$ (i.e. an eigenvector for eigenvalue $\lambda = 1$). You can do this without any computation; think geometrically about T.



$\boxed{\vec{u} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}}$

(since $\vec{u}$ lies on the axis of reflection $y=8x$).

(b) Find a nonzero vector $\vec{v}$ such that $T(\vec{v}) = -\vec{v}$ (in other words, an eigenvector for eigenvalue $\lambda = -1$). Again, you can do this without much computation; it's useful to think about the dot product with $\vec{u}$.



$\boxed{\vec{v} = \begin{pmatrix} -8 \\ 1 \end{pmatrix}}$

(since $\vec{v} \perp \vec{u}$, it is perpendicular to the axis of reflection & thus $T(\vec{v}) = -\vec{v}$).

(continued on reverse)

(c) Let $B = \{\vec{u}, \vec{v}\}$, where these are the vectors you found in parts (a), (b). This is a basis for $\mathbb{R}^2$ (you don't need to prove this). Find $[T]_B$ (this should not require any computations at all; just use the equations $T(\vec{u}) = \vec{u}$ and $T(\vec{v}) = -\vec{v}$).

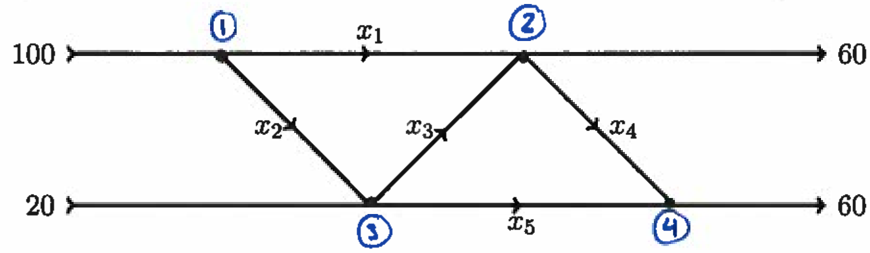
$$B = \{\vec{u}, \vec{v}\}$$
$$[T(\vec{u})]_B = [\vec{u}]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\& \quad [T(\vec{v})]_B = [-\vec{v}]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$
hence $\boxed{[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

(d) Compute $[I]_B^S$, and use it to compute $[T]_S$ (here, S is the standard basis for $\mathbb{R}^2$).

$$[I]_B^S = \begin{pmatrix} 1 & -8 \\ 8 & 1 \end{pmatrix}, \quad \text{so}$$

$$[T]_S = \begin{pmatrix} 1 & -8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -8 \\ 8 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & -8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{65}\begin{pmatrix} 1 & 8 \\ -8 & 1 \end{pmatrix}$$
$$= \frac{1}{65}\begin{pmatrix} 1 & -8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 8 \\ 8 & -1 \end{pmatrix}$$
$$= \boxed{\frac{1}{65}\begin{pmatrix} -63 & 16 \\ 16 & 63 \end{pmatrix}} \quad \left(\text{or } \begin{pmatrix} -63/65 & 16/65 \\ 16/65 & 63/65 \end{pmatrix}\right).$$

8. [9 points] Consider the traffic flow pattern shown below. The diagram is interpreted as follows: 100 cars enter along the inbound road in the upper left, and 20 cars enter along the road in the lower left. Along both of the outbound roads on the right, 60 cars exit. The traffic aong the five road segments in the middle are denoted by variables x_1, x_2, x_3, x_4, x_5.



(a) Write a system of linear equation in x_1, x_2, x_3, x_4, x_5 describing the traffic flow in this network.

flow in = flow out.

one eq'n per intersection:

$$\begin{cases} ① & 100 = x_1 + x_2 \\ ② & x_1 + x_3 = x_4 + 60 \\ ③ & 20 + x_2 = x_3 + x_5 \\ ④ & x_4 + x_5 = 60 \end{cases}$$

(b) Find the general solution to this system of linear equations.
Hint: your answer should involve two free variables.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 1 & 0 & 1 & -1 & 0 & 60 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right) \xrightarrow{R2 -= R1} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & -1 & 1 & -1 & 0 & -40 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right)$$

$$\xrightarrow[R2 *= (-1)]{R1, R3 += R2} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 60+60 \\ 0 & 1 & -1 & 1 & 0 & 40 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right)$$

$$\boxed{\begin{aligned} x_1 &= 120 - x_3 - x_5 \\ x_2 &= -20 + x_3 + x_5 \\ x_3 &\ \text{free} \\ x_4 &= 60 - x_5 \\ x_5 &\ \text{free} \end{aligned}}$$

$$\xrightarrow[\substack{R4 += R3 \\ R3 *= (-1)}]{\substack{R1 -= R3 \\ R2 += R3}} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 120 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(continued on reverse)

Additional space for part (b)

(c) You found in part (b) that there are infinitely many possible traffic patterns in this network. In reality, some patterns are more plausible than others. For example, drivers will tend to drive on less busy roads, causing the traffic to balance itself across the various roads.

One way to model this is to assume that the drivers will choose the traffic pattern than minimizes the quantity

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2.$$

Among all possible solutions found in part (b), determine which solution minimizes this quantity. Your answer should be a specific choice of values x_1, x_2, x_3, x_4, x_5.

Hint: The quantity to be minimized is the same as $\|\vec{x}\|^2$, and you have expressed $\vec{x}$ in terms of two free variables. You can convert this into a least-squares problem.

gen'l sol'n $$\vec{X} = \begin{pmatrix} 120 \\ -20 \\ 0 \\ 60 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

so we wish to minimize

$$\left\| x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -120 \\ 20 \\ 0 \\ -60 \\ 0 \end{pmatrix} \right\|^2.$$

Let $\vec{u} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -120 \\ 20 \\ 0 \\ -60 \\ 0 \end{pmatrix}$.

By the normal eq'n, $\|x_3\vec{u} + x_5\vec{v} - \vec{b}\|^2$ is minimized when

$$\begin{pmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v} \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} \vec{u}\cdot\vec{b} \\ \vec{v}\cdot\vec{b} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1+1+1 & 1+1 \\ 1+1 & 1+1+1+1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 120+20+0+0+0 \\ 120+20+0+60+0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 140 \\ 200 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 140 \\ 200 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 140 \\ 200 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 560 - 400 \\ -280 + 600 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 160 \\ 320 \end{pmatrix}$$

$$= \begin{pmatrix} 20 \\ 40 \end{pmatrix}.$$

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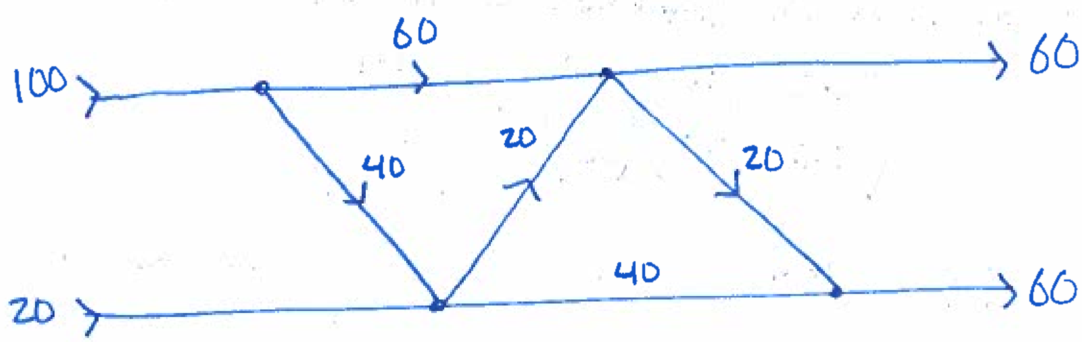
So the optimum sol'n is

$$\begin{pmatrix} 120 \\ -20 \\ 0 \\ 60 \\ 0 \end{pmatrix} + 20 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 40 \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 120 & -20 & -40 \\ -20 & +20 & +40 \\ 0 & +20 & +0 \\ 60 & +0 & -40 \\ 0 & +0 & +40 \end{pmatrix} = \boxed{\begin{pmatrix} 60 \\ 40 \\ 20 \\ 20 \\ 40 \end{pmatrix}}$$

This pattern is shown in the figure below.



Remark This optimization problem is the same as a circuit problem, where all four roads have equal "resistance". You may enjoy working out why this is!

9. [9 points] Let $T : V \to W$ be a linear transformation.

(a) Prove that if T is injective, then $N(T) = \{\vec{0}\}$.

Suppose that $\vec{v} \in N(T)$.

Then $T(\vec{v}) = \vec{0}$ (def'n of $N(T)$).

But $T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}$ as well.

Since T is injective, $T(\vec{v}) = T(\vec{0})$ implies $\vec{v} = \vec{0}$.

So the only vector in $N(T)$ is $\vec{0}$ itself.

(b) Prove the converse: if $N(T) = \{\vec{0}\}$, then T is injective.

Suppose that $T(\vec{x}) = T(\vec{y})$.

Then $T(\vec{x}) - T(\vec{y}) = \vec{0}$

$\Rightarrow T(\vec{x} - \vec{y}) = \vec{0}$ (T is linear)

$\Rightarrow \vec{x} - \vec{y} \in N(T)$ (def'n of $N(T)$)

$\Rightarrow \vec{x} - \vec{y} = \vec{0}$ (assumption: $N(T) = \{\vec{0}\}$)

$\Rightarrow \vec{x} = \vec{y}$.

So $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$. This means that T is injective.

(continued on reverse)

(c) Prove that if T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$, then T cannot be injective.

By the rank-nullity theorem,

$$\dim N(T) = \dim \mathbb{R}^4 - \dim R(T)$$
$$= 4 - \dim R(T).$$

Since $R(T)$ is a subspace of $\mathbb{R}^3$,

$$\dim R(T) \leq \dim \mathbb{R}^3 = 3.$$

So $\dim N(T) = 4 - \dim R(T) \geq 4 - 3 = 1.$

This means $N(T)$ has nonzero vectors in it.

By part (a), T is not injective.

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