Note: This assignment covers two weeks of material, to avoid having an assignment due in an exam week. Therefore it is longer than usual.

Textbook problems from DeFranza and Gagliardi:

Suggestion: Do the odd-numbered problems from 1 to 15 in §4.1, and check your answer in the back of the book. Try to develop a good intuition for what is linear and what is not.

- §4.1: 22, 26, 30, 39, 42, 44
- §4.2: 6, 16, 22, 32, 38, 40

Midterm 2 material ends here

- §4.3: 8, 18, 28, 30
- §4.4: 14, 20, 30, 36

Supplemental problems:

Any supplemental problems will not concern material on the exam, and will be posted after it.

1. (Polynomial interpolation). This sequence of problems concerns the following situation: given a list of data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), is it possible to fit a polynomial equation \(y = p(x)\) of degree \(d\) to these points?

   (a) Suppose that \(n\) distinct numbers \(x_1, \ldots, x_n\) are chosen, and consider the linear transformation

   \[ T(p(x)) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_n) \end{pmatrix} \]

   given by \(T(p(x))\). Prove that \(T\) is an isomorphism.

   Hint: you may use, without proof, the following fact from algebra: if \(p(x)\) is a nonzero polynomial of degree \(d\), then \(p(x) = 0\) can have at most \(d\) solutions.

   (b) Deduce from (a) that given a list of \(n\) data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), where all of the \(x\)-coordinates are distinct, there is always a unique polynomial \(p(x)\) of degree \(n - 1\) passing through all of them (i.e. \(p(x_i) = y_i\) for each \(i\)).

   (c) Find the unique cubic (degree 3) polynomial \(p(x)\) such that \(y = p(x)\) passes through \((-1, -11), (0, -7), (1, -3)\), and \((2, 7)\).

2. Consider the transformation \(T : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) given by reflection across the line \(y = 7x\).

   (a) Find a basis \(B\) in which the matrix representation of \(T\) is \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

   (b) Use your answer from (a) to determine the matrix representation of \(T\) in the standard basis.

(Same “important notes” and “submission instructions” apply as before; omitted to save space.)