Textbook problems from DeFranza and Gagliardi:

- §2.3: 6, 8, 36, 42
- §6.1: 14, 18, 23, 32
- §3.2: 18, 20, 24, 26, 30, 44, 50

Supplemental problems:

Note: You may wish to use Mathematica, or other software, for the computations in some of the problems below. If you do, clearly indicate which computations you are using software for, or print out and include the Mathematica input/output with your submission.

1. A square matrix A is called *orthogonal* if $A^tA = I$ (in other words: if it is invertible, and its inverse is equal to its transpose). Prove that if A is orthogonal, then $\det A$ is either 1 or -1.

2. Find the linear combination of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ that is closest to $\begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}$.

3. Define a matrix A and vector $\vec{b}$ as follows.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 8 \\ 4 \\ 12 \\ 0 \end{pmatrix}$$

   (a) Verify that the linear system $A\vec{x} = \vec{b}$ is inconsistent.
   (b) Find the "least-squares" solution, i.e. the vector $\vec{x}$ which minimizes $\|A\vec{x} - \vec{b}\|$.

4. Consider the following four points in the plane. This problem will demonstrate a couple ways that we could find a "line of best fit" for these four points. Part (a) is the usual method. The purpose of this exercise is to see how choosing a different "objective" (function to be minimized) can product different "lines of best fit" (essentially because it depends on what "best" means, which may be different in different applications).

$$(x_1, y_1) = (1, 1) \qquad (x_2, y_2) = (3, 2) \qquad (x_3, y_3) = (1, 6) \qquad (x_4, y_4) = (3, 7)$$

   (a) Suppose that we wish to find the coefficients c_1, c_2 that minimize the sum $\sum_{i=1}^{4}(c_1\, x_i + c_2 - y_i)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2. Sketch the four points and the line $y = c_1x + c_2$.
   (b) Suppose that we now want coefficients c_1, c_2 that minimize $\sum_{i=1}^{4}(c_1y_i + c_2 - x_i)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2 and sketch the four points and the line $x = c_1y + c_2$.
   (c) A third way to specify a line is using an equation of the form $c_1x + c_2y = 1$. Suppose that now we wish to find c_1, c_2 minimizing $\sum_{i=1}^{4}(c_1x_i + c_2y_i - 1)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2 and sketch the four points with the line $c_1x + c_2y = 1$.

(Same "important notes" and "submission instructions" apply as before; they will be omitted from now to reduce clutter.)