supplement 1

a) $T = \begin{pmatrix} 0.6 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$

$$\det(T-\lambda I) = \begin{vmatrix} 0.6-\lambda & 0.3 & 0.4 \\ 0.1 & 0.4-\lambda & 0.3 \\ 0.3 & 0.3 & 0.3-\lambda \end{vmatrix}$$

$$= (0.6-\lambda)(0.4-\lambda)(0.3-\lambda) + 0.3\cdot 0.3\cdot 0.3 + 0.4\cdot 0.1\cdot 0.3$$
$$- (0.6-\lambda)\cdot 0.3\cdot 0.3 - 0.3\cdot 0.1\cdot(0.3-\lambda) - 0.4\cdot(0.4-\lambda)\cdot 0.3$$

$$= 0.072 - 0.54\lambda + 1.3\lambda^2 - \lambda^3$$
$$+ 0.027 + 0.012$$
$$- 0.054 + 0.09\lambda$$
$$- 0.009 + 0.03\lambda$$
$$- 0.048 + 0.12\lambda$$

$$= -\lambda^3 + 1.3\lambda^2 - 0.3\lambda$$
$$= -\lambda(\lambda-0.3)(\lambda-1)$$

So the eigenvalues of T are $1, 0.3, 0$. To find the eigenspaces:

$\lambda_1 = 1$
$$V_1 = N\begin{pmatrix} -0.4 & 0.3 & 0.4 \\ 0.1 & -0.6 & 0.3 \\ 0.3 & 0.3 & -0.7 \end{pmatrix} \quad \text{(null-space)}$$
$$= N\begin{pmatrix} 1 & 0 & -11/7 \\ 0 & 1 & -16/21 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(row-reduction)}$$
$$= \operatorname{span}\left(\begin{bmatrix} 33 \\ 16 \\ 21 \end{bmatrix}\right); \quad \text{let } \underline{\vec{v}_1 = \begin{pmatrix} 33 \\ 16 \\ 21 \end{pmatrix}}.$$

$\lambda_2 = 0.3$
$$V_{0.3} = N\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0.1 & 0.3 \\ 0.3 & 0.3 & 0 \end{pmatrix}$$
$$= N\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right); \quad \text{let } \underline{\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda_3 = 0$
$$V_0 = N\begin{pmatrix} 0.6 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$
$$= N\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}; \quad \text{let } \underline{\vec{v}_3 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}}.$$

So one possible eigenbasis is $B = \left\{ \begin{pmatrix} 33 \\ 16 \\ 21 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \right\}$.

So the change of basis matrix P is

$$\underline{P = [I]^{S}_{B} = \begin{pmatrix} 33 & -1 & -1 \\ 16 & 1 & -2 \\ 21 & 0 & 3 \end{pmatrix}},$$

which has inverse

$$P^{-1} = \frac{1}{210} \cdot \begin{pmatrix} 3 & 3 & 3 \\ -90 & 120 & 50 \\ -21 & -21 & 49 \end{pmatrix},$$ (computations omitted)

and the diagonal form of T is

$$\underline{D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0 \end{pmatrix}}. \quad (\underline{A = PDP^{-1}}).$$

b) Hence the explicit form for T^n is $P \cdot D^n \cdot P^{-1}$, ie.

$$T^n = \begin{pmatrix} 33 & -1 & -1 \\ 16 & 1 & -2 \\ 21 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & (0.3)^n & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{210} \cdot \begin{pmatrix} 3 & 3 & 3 \\ -90 & 120 & 50 \\ -21 & -21 & 49 \end{pmatrix}$$

$$= \frac{1}{210} \cdot \begin{pmatrix} 33 & -(0.3)^n & 0 \\ 16 & (0.3)^n & 0 \\ 21 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 & 3 \\ -90 & 120 & 50 \\ -21 & -21 & 49 \end{pmatrix}$$

$$= \boxed{\frac{1}{210} \cdot \begin{pmatrix} 99 + 90(0.3)^n & 99 - 120 \cdot (0.3)^n & 99 - 50(0.3)^n \\ 48 - 90 \cdot (0.3)^n & 48 + 120 \cdot (0.3)^n & 48 + 50 \cdot (0.3)^n \\ 63 & 63 & 63 \end{pmatrix}}.$$

Hence we may check the answer to part (b) of 5.4.4 as follows:

prob. after 5 fares: $T^5 \cdot \begin{pmatrix} 0.3 \\ 0.35 \\ 0.35 \end{pmatrix}$

$$= \frac{1}{210} \begin{pmatrix} 99+90(0.3)^5 & 99-120(0.3)^5 & 99-50\cdot(0.3)^5 \\ 48-90(0.3)^5 & 48+120\cdot(0.3)^5 & 48+50(0.3)^5 \\ 63 & 63 & 63 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.35 \\ 0.35 \end{pmatrix}$$

$$= \frac{1}{210} \begin{pmatrix} 99 - 32.5\cdot(0.3)^5 \\ 48 + 32.5\cdot(0.3)^5 \\ 63 \end{pmatrix}$$

$$= \frac{1}{210} \cdot \begin{pmatrix} 98.921 \\ 48.079 \\ 63 \end{pmatrix}$$

$$= \begin{pmatrix} 0.471053. \\ 0.228948. \\ 0.3 \end{pmatrix}.$$

c) In $\lim_{n\to\infty} T^n$, all of the terms w/ $(0.3)^n$ go to 0.

Hence

$$\lim_{n\to\infty} T^n = \frac{1}{210} \begin{pmatrix} 99 & 99 & 99 \\ 48 & 48 & 48 \\ 63 & 63 & 63 \end{pmatrix}$$

$$= \begin{pmatrix} 33/70 & 33/70 & 33/70 \\ 16/70 & 16/70 & 16/70 \\ 21/70 & 21/70 & 21/70 \end{pmatrix}.$$

(3)

Imitating 2(a),

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix} \qquad (\text{since } G_{n+1} = G_n + 2G_{n-1})$$

$$\Rightarrow \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} G_1 \\ G_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So G_n is the lower-left entry of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n$.

To diagonalize $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$:

$$0 = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1)$$

so $\lambda_1 = 2$, $\lambda_2 = -1$.

For $\lambda_1 = 2$: nullspace $\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} = \text{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

so let $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -1$, nullspace $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

so let $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Hence $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = PDP^{-1}$ where

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

Hence

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2\cdot 2^n & -(-1)^n \\ 2^n & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2\cdot 2^n + (-1)^n & 2\cdot 2^n - 2(-1)^n \\ 2^n - (-1)^n & 2^n + 2(-1)^n \end{pmatrix}.$$

Since G_n is the lower-left entry, we obtain

$$\boxed{G_n = \frac{1}{3}\left(2^n - (-1)^n\right).}$$