1

a)

$\mathbb{R}^3$ $\quad \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

$M_{2\times2}$ $\quad \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$

P_2 $\quad \{1, x, x^2\}$

(many other answers are possible, of course).

b) $$[\vec{v}]_{B'} = \begin{pmatrix}3&-4\\4&3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \boxed{\begin{pmatrix}-1\\7\end{pmatrix}}.$$

c) $$\dim R(T) = \dim \mathbb{R}^3 - \dim N(T)$$
$$= 3-1 = \boxed{2}.$$

(2)

$$B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

$$B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} \right]_{B'} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{by solving: } \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array} \right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right)$$

$$\& \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]_{B'} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{by solving: } \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \end{array} \right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right)$$

Hence $[I]_B^{B'} = \boxed{\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}}$

3 a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow[R4 -= 2R1]{R2 -= 2R1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R1 -= 2R2]{R2 \leftrightarrow R3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(RREF)}$$

Pivots in columns 1&2, so $\boxed{\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix} \right\}}$ form a basis of W.

(ie. $\{\vec{u}, \vec{v}\}$).

b) $\dim W = 2$ (two elements in the basis found in (a)).

c) Gram-Schmidt applied to $\vec{u}, \vec{v}$:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{2+8+0+8}{1+4+0+4} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 4 \end{pmatrix}$$

$$\vec{v} - \text{proj}_{\vec{u}}(\vec{v}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

so $\left\{ \vec{u}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ form an <u>orthogonal</u> basis.

For an orthonormal basis, normalize both:

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{1+4+0+4}} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}$$

$$\frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\boxed{\left\{ \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}}$$

(other answers possible, of course).

d) There are two approaches here that we've discussed in this class (& many others using other techniques).

<u>method 1</u> : least-squares:

W has basis $\vec{u}, \vec{v}$, so we can optimize

$$\|c_1\vec{u} + c_2\vec{v} - \vec{b}\|$$

by solving

$$\begin{pmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v} \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{u}\cdot\vec{b} \\ \vec{v}\cdot\vec{b} \end{pmatrix}$$

$$\begin{pmatrix} 9 & 18 \\ 18 & 37 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 9 & 18 & 5 \\ 18 & 37 & 11 \end{array}\right) \to \left(\begin{array}{cc|c} 9 & 18 & 5 \\ 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 9 & 0 & -13 \\ 0 & 1 & 1 \end{array}\right)$$

so $c_1 = -\frac{13}{9}$, $c_2 = 1$ is optimal. The closest point is

$$-\frac{13}{9}\begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5/9 \\ 10/9 \\ 1 \\ 10/9 \end{pmatrix}}$$

<u>method 2</u> The special-purpose approximation method for orthogonal sets (see notes on approx. in inner product spaces):

Use the orthonormal basis from (c). Denote the vectors by $\vec{w}_1, \vec{w}_2$ for convenience:

$$\vec{w}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The nearest combination to $\vec{b}$ is

$$\text{proj}_{\vec{w}_1}(\vec{b}) + \text{proj}_{\vec{w}_2}(\vec{b})$$

$$= \frac{\vec{w}_1\cdot\vec{b}}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \frac{w_2\cdot\vec{b}}{\vec{w}_2\cdot\vec{w}_2}\vec{w}_2$$

$$= \frac{\frac{1}{3}+\frac{2}{3}+0+\frac{2}{3}}{1}\cdot\vec{w}_1 + \frac{1}{1}\vec{w}_2$$

$$= \frac{5}{3}\cdot\begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5/9 \\ 10/9 \\ 1 \\ 10/9 \end{pmatrix}}$$

(4)

Since norms are nonnegative (positive-definiteness), it's equivalent to prove that

$$\| \vec{u} + 2\vec{v} \|^2 > \|\vec{u}\|^2. \quad \text{(square both sides).}$$

Viewing the squared norm as an inner product, observe that

$$\|\vec{u}+2\vec{v}\|^2 = \langle \vec{u}+2\vec{v}, \vec{u}+2\vec{v} \rangle$$

$$= \langle \vec{u}+2\vec{v}, \vec{u} \rangle + 2\langle \vec{u}+2\vec{v}, \vec{v} \rangle \quad \text{(linearity in 2nd argument)}$$

$$= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{v}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle + 4\langle \vec{v}, \vec{v} \rangle \quad \text{(linearity in 1st argument)}$$

$$= \langle \vec{u}, \vec{u} \rangle + 4\langle \vec{v}, \vec{v} \rangle \quad \text{(since } \langle \vec{u}, \vec{v} \rangle = 0, \text{ \& hence } \langle \vec{v}, \vec{u} \rangle = 0 \text{ also, by symmetry)}$$

$$= \|\vec{u}\|^2 + 4\|\vec{v}\|^2$$

$$> \|u\|^2 \quad \text{(since } \vec{v} \text{ is nonzero, } \|\vec{v}\| > 0 \text{ by positive definiteness).}$$

So indeed $\|\vec{u}+2\vec{v}\|^2 > \|\vec{u}\|^2$, as desired.

(5) Row-reducing:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow[R4 -= 2R2]{R3 -= R1,\; R3 -= R2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R1 -= R2} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Pivots in columns 1 & 3, so $\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\}}$ ~~are~~ is a basis for $R(T)$.

(the corresponding columns from the original matrix).

b) using the RREF, the general sol'n to $A\vec{x} = \vec{0}$ can be written

$$\begin{pmatrix} -2x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \quad (x_2, x_4 \text{ free})$$

$$= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

so $\boxed{\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$ is a basis for $N(T)$.

⑥ a)

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx$$ defines an inner product on $\mathcal{C}[-\pi,\pi]$.

So the inner products given amount to:

$$\langle f, \sin x\rangle = 7$$

$$\langle f, \cos x\rangle = 13$$

$$\langle \sin x, \sin x\rangle = \pi$$

$$\langle \sin x, \cos x\rangle = 0$$

$$\langle \cos x, \cos x\rangle = \pi.$$

b) Since $\sin x \perp \cos x$, the optimum choices are:

$$c_1 = \frac{\langle f, \sin x\rangle}{\langle \sin x, \sin x\rangle} = \frac{7}{\pi} \quad (\text{ensures } (c_1 \sin x + c_2 \cos x - f(x)) \perp \sin x)$$

$$c_2 = \frac{\langle f, \cos x\rangle}{\langle \cos x, \cos x\rangle} = \frac{13}{\pi} \quad (\text{ensures } (c_1 \sin x + c_2 \cos x - f(x)) \perp \cos x).$$

$$\boxed{c_1 = \frac{7}{\pi}, \quad c_2 = \frac{13}{\pi}.}$$