1. [9 points] Solve the following system of linear equations.

$$\begin{array}{rcrcrcrcr} & & x_2 & + & x_3 & + & x_4 & = & 5 \\ x_1 & & & + & 3x_3 & + & 7x_4 & = & 14 \\ x_1 & & & + & 2x_3 & + & 5x_4 & = & 11 \end{array}$$

Row-reducing the aug. matrix:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 5 \\ 1 & 0 & 3 & 7 & 14 \\ 1 & 0 & 2 & 5 & 11 \end{array}\right] \quad -1\times$$

$$\downarrow$$

$$+3\times \quad +1\times \left[\begin{array}{cccc|c} 1 & 0 & 3 & 7 & 14 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & -1 & -2 & -3 \end{array}\right] \times(-1)$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$\underbrace{}_{x_1, x_2, x_3 \text{ bound}}$ $\underbrace{}_{x_4 \text{ free}}$

$$\boxed{\begin{aligned} x_1 &= 5 - t \\ x_2 &= 2 + t \\ x_3 &= 3 - 2t \\ x_4 &= t \end{aligned}}$$

2. [9 points]
Recall that two matrices A, B *commute* if $AB = BA$. Consider the following three matrices.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

(a) Determine whether A and B commute.

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\neq$$
$$BA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$\boxed{\text{no}}$

(b) Determine whether B and C commute.

$$BC = \begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$$
$$=$$
$$CB = \begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$$

$\boxed{\text{yes}}$

(c) Determine whether A and C commute.

$$AC = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$
$$\neq$$
$$CA = \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$$

$\boxed{\text{no}}$

3. [9 points]

(a) Suppose that A is an $n \times n$ matrix. Show that if A is invertible, then $A\vec{x} = \vec{0}$ has no nontrivial solutions $\vec{x}$.

Given A^{-1} exists:

If $A\vec{x} = \vec{0}$

then $A^{-1}A\vec{x} = A^{-1}\vec{0}$

$\Rightarrow \vec{x} = \vec{0}$. (since $A^{-1}A = I$ & $A^{-1}\vec{0} = \vec{0}$).

So any sol'n is the trivial one. i.e. there are no nontrivial sol'ns.

(b) Using part (a), show that $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ -1 & -1 & 2 \end{pmatrix}$ is not an invertible matrix. (*Hint:* the numbers in each row sum to 0.)

$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \vec{A}_1 + 1 \cdot \vec{A}_2 + 1 \cdot \vec{A}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So $A\vec{x} = \vec{0}$ has a nontrivial sol'n, namely $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

By (a), this is impossible if A is invertible.

So A must not be invertible.

4. [9 points] The augmented matrix of a linear system has the form

$$\begin{array}{c} -2\times \\ +1\times \\ \\ \end{array}\left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 2 & 3 & -2 & b \\ -1 & -1 & 1 & c \end{array}\right).$$

(a) Determine the values of a, b, c for which this linear system in consistent.

row-reducing gives

$$\begin{array}{c} +2\times \\ +1\times \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 0 & b-2a \\ 0 & 1 & 0 & a+c \end{array}\right]\begin{array}{c} \\ \times(-1) \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3a+2b \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & -a+b+c \end{array}\right]$$

Which is consistent iff the last row is all 0's. i.e. $\boxed{a=b+c}$.

(b) For those values of a, b, c for which the system is consistent, does it have a unique solution or infinitely many solutions? Briefly explain why.

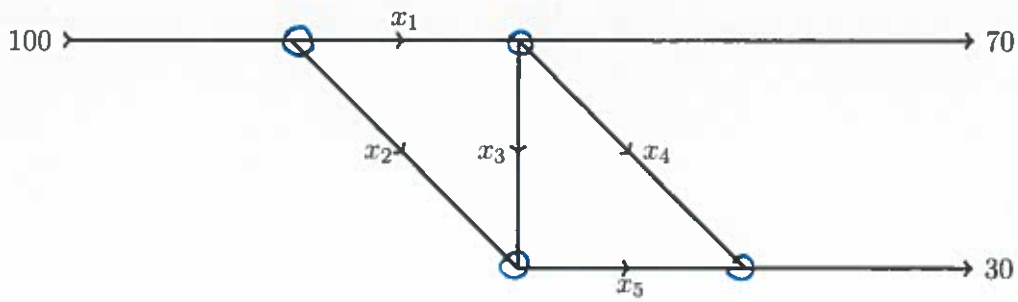
There are ∞ sol'ns (when consistent) because x_3 will be a free variable (no pivot in column 3).

(In fact, the sol'n is

$$x_1 = -3a+2b+t$$
$$x_2 = 2a-b$$
$$x_3 = t$$

in this case).

5. [9 points] Write a system of linear equations that describes the traffic flow pattern for the network in the figure. **You do not need to solve the system.**



traffic in = traffic out

$$\begin{cases} 100 = x_1 + x_2 \\ x_1 = x_3 + x_4 + 70 \\ x_2 + x_3 = x_5 \\ x_4 + x_5 = 30 \end{cases}$$

or, written in the usual way,

$$\begin{cases} x_1 + x_2 = 100 \\ x_1 - x_3 - x_4 = 70 \\ x_2 + x_3 - x_5 = 0 \\ x_4 + x_5 = 30 \end{cases}$$

2. [9 points] Consider the following three vectors.

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

(a) Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

row-reduce:

Note: this was self-check problem 2.3.23.

$$\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix} \longrightarrow \begin{bmatrix}1&1&1\\0&1&0\\0&2&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Pivot in each column $\Rightarrow$ sol'ns to $\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix}\vec{x} = \vec{0}$ have no free variables $\Rightarrow$ columns are lin. indep.

(b) Find the unique scalars c_1, c_2, c_3 such that the vector

$$\vec{v} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

is equal to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

same row ops., but w/ the aug. matrix:

$$\left[\begin{array}{ccc|c}1&1&1&2\\1&2&1&1\\1&3&2&3\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&1&1&2\\0&1&0&-1\\0&2&1&1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&-1\\0&0&1&3\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1\\0&0&1&3\end{array}\right].$$

$$\boxed{c_1 = 0, \quad c_2 = -1, \quad c_3 = 3}$$