Consider the matrix $A = \begin{pmatrix} 5 & -1 \\ 12 & -2 \end{pmatrix}$. (Same A in all problems).

① What are the eigenvalues of A?

$$0 = \begin{vmatrix} 5-\lambda & -1 \\ 12 & -2-\lambda \end{vmatrix} = (5-\lambda)(-2-\lambda) + 12 = \lambda^2 - 3\lambda + 2$$
$$= (\lambda-1)(\lambda-2)$$

$$\boxed{\lambda = 1 \ \& \ \lambda = 2}$$

② Find an eigenvector for each eigenvalue.

$\underline{\lambda_1 = 1}$: $\begin{pmatrix} 5-1 & -1 \\ 12 & -2-1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 12 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1/4 \\ 0 & 0 \end{pmatrix}$ nullspace $\begin{pmatrix} t/4 \\ t \end{pmatrix}$

eg. $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}}$ spans. (check: $\begin{pmatrix} 5 & -1 \\ 12 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$).

$\underline{\lambda_2 = 2}$ $\begin{pmatrix} 5-2 & -1 \\ 12 & -2-2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 12 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1/3 \\ 0 & 0 \end{pmatrix}$ nullspace $\begin{pmatrix} t/3 \\ t \end{pmatrix}$

eg. $\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$ spans. (check: $\begin{pmatrix} 5 & -1 \\ 12 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$).

③ Find matrices P and D such that $\underline{D \text{ is diagonal}}$ and $\underline{A = PDP^{-1}}$ (You don't need to compute P^{-1}, but it may be helpful to do so to check your answer).

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$
$= \{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \}$
(eigenbasis)

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}$ (A in basis $\vec{v}_1, \vec{v}_2$)

$P = [I]^{S}_{\mathcal{B}} = \boxed{\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}}$. (columns are $\vec{v}_1, \vec{v}_2$).

NOTE swapping the cols of P, or multiplying them by constants, gives other equally valid answers.

So: $\underline{A = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}^{-1}}$

$\underline{\text{Note}}$: $\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$; the answers may be checked by computing that $\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 12 & 2 \end{pmatrix}$.

An explicit formula for ~~the~~ A^n $\left(A = \begin{pmatrix} 5 & -1 \\ 12 & -2 \end{pmatrix}\right.$, the matrix from the quiz).

We saw on the quiz that:

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}^{-1}.$$

It follows that, taking n^{th} powers:

$$A^n = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix}}_{\text{eigenvalues raised to } n^{th} \text{ power}} \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 4\cdot 2^n & -2^n \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} 5 & -1 \\ 12 & -2 \end{pmatrix}^n = \begin{pmatrix} -3+4\cdot 2^n & 1-2^n \\ -12+12\cdot 2^n & 4-3\cdot 2^n \end{pmatrix}}$$

(try plugging in $n=0,1,2$ to check this).

So we obtain an explicit form for the entries of A^n.