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Proofs of claims from the "review of key concepts" handout

Note: following our in-class convention since §3.3, I'll use $+, \cdot$ instead of $\oplus, \odot$ below.

Claim 1 Any subspace W of a vector space V contains $\vec{0}$.

Pf W is closed under scalar multiplication. So for every $\vec{w} \in W$, $0 \cdot \vec{w} \in W$ as well. But $0 \cdot \vec{v} = \vec{0}$ for all $\vec{v} \in V$.

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see Thm. 9, pg. 159.

Claim 2 A list of vectors $\vec{v}_1, \cdots, \vec{v}_n \in V$ is linearly independent if and only if $\forall \vec{v} \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$, there are unique coefficients $c_1, \cdots, c_n \in \mathbb{R}$ s.t. $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

Pf 1) Suppose $\vec{v}_1, \cdots, \vec{v}_n$ are lin. indep.. Suppose also that $\vec{v}$ is in their span. So $\exists$ constants $c_1, \cdots, c_n$ st $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. To see that they are unique, suppose that

$$\vec{v} = c_1'\vec{v}_1 + \cdots + c_n'\vec{v}_n$$

for some list of constants $c_1', \cdots, c_n'$; we will show that in fact $c_1' = c_1$, $c_2' = c_2, \cdots$, $c_n' = c_n$. This will establish that there is no other way to choose the constants.

Indeed:
$$c_1'\vec{v}_1 + c_2'\vec{v}_2 + \cdots + c_n'\vec{v}_n = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$\Rightarrow (c_1'\vec{v}_1 - c_1\vec{v}_1) + (c_2'\vec{v}_2 - c_2\vec{v}_2) + \cdots + (c_n'\vec{v}_n - c_n\vec{v}_n) = \vec{0}$$

$$\Rightarrow (c_1' - c_1)\vec{v}_1 + (c_2' - c_2)\vec{v}_2 + \cdots + (c_n' - c_n)\vec{v}_n = \vec{0}$$

$$\Rightarrow c_1' - c_1 = c_2' - c_2 = \cdots = c_n' - c_n = 0 \quad \text{(by lin. independence)}$$

$$\Rightarrow c_1' = c_1, \quad c_2' = c_2, \quad \cdots, \quad c_n' = c_n,$$

as claimed.

2) For the converse, suppose that all elements of the span have unique coefficients. Then $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_n$ must be the unique way to write $\vec{0}$ as a lin comb of $\vec{v}_1, \cdots, \vec{v}_n$. So the only way to write

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

is if $c_1 = c_2 = \cdots = c_n = 0$. Hence the list $\vec{v}_1, \cdots, \vec{v}_n$ is lin. indep.

Claim 3 If $B = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\forall \vec{v} \in V$, there are unique "coordinates" $c_1, \cdots, c_n \in \mathbb{R}$ st. $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

Pf ~~This~~ Since B is a basis, span$\{\vec{v}_1, \cdots, \vec{v}_n\} = V$ and $\vec{v}_1, \cdots, \vec{v}_n$ are lin indep. So by claim 2, all $\vec{v} \in V$ ($=$ span$\{\vec{v}_1, \cdots, \vec{v}_n\}$) determine unique constants $c_1, \cdots, c_n$ st. $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

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see Thm. 5 on pg. 115.

Claim 4 $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is lin. indep. iff no vector in the list is in the span of the others

Pf 1) Suppose that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is lin. indep. Suppose $\lightning$ (for contradiction) that one of them, say $\vec{v}_i$, is in the span of the others. This means that

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n.$$

for some constants $c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n$.

This implies that

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n,$$

contradicting the linear independence of $\vec{v}_1, \cdots, \vec{v}_n$. $\lightning$

So none of $\vec{v}_1, \cdots, \vec{v}_n$ is a lin. comb. of the others.

2) Conversely, suppose that no $\vec{v}_i$ is a lin. comb. of the others.

↯ Suppose (for contradiction) that $\vec{v}_1, \cdots, \vec{v}_n$ are linearly dependent. Then $\exists c_1, \cdots, c_n \in \mathbb{R}$, not all 0, st

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$

Suppose $c_i \neq 0$ (we've assumed at least one of the c_i isn't 0, so choose one). Then:

$$-c_i\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n$$

(mult. by $-\frac{1}{c_i}$) $\Rightarrow \vec{v}_i = \left(-\frac{c_1}{c_i}\right)\vec{v}_1 + \left(-\frac{c_2}{c_i}\right)\vec{v}_2$

$$+ \cdots + \left(-\frac{c_{i-1}}{c_i}\right)\vec{v}_{i-1} + \left(-\frac{c_{i+1}}{c_i}\right)\vec{v}_i$$

$$+ \cdots + \left(-\frac{c_n}{c_i}\right)\vec{v}_n$$

So $\vec{v}_i$ is a lin. comb. of the others, a contradiction. ↯.

So $\vec{v}_1, \cdots, \vec{v}_n$ are in fact lin. indep.

<u>Claim 5</u> If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is lin. <u>dependent</u>, then any $\vec{v}$ in their span can be written

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

in infinitely many ways.

<u>Proof</u> There are constants $d_1, \cdots, d_n \in \mathbb{R}$, not all 0, st.

$$\vec{0} = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n.$$

Hence whenever $\vec{v}$ can be written

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n,$$

it follows that $\forall t \in \mathbb{R}$,

$$\begin{aligned}\vec{v} &= \vec{v} + t\cdot\vec{0} \\ &= (c_1 + t\cdot d_1)\cdot\vec{v}_1 + (c_2 + t\cdot d_2)\cdot\vec{v}_2 + \cdots + (c_n + t\cdot d_n)\vec{v}_n.\end{aligned}$$

Since not all of $d_1, \cdots, d_n$ are 0, the list of coefficients

$$c_1 + t \cdot d_1, \quad \cdots, \quad c_n + t \cdot d_n$$

is different for every choice of t (more precisely: if $d_i \neq 0$, then $c_i + t \cdot d_i = c_i + t' d_i$ iff $t = t'$, so the coeff. $c_i + t \cdot d_i$ is different for every value of $t \in \mathbb{R}$).

So $\vec{v}$ is a lin. comb. of $\vec{v}_1, \cdots, \vec{v}_n$ in ∞ ways.

<u>Claim 6</u> $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\} = \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ iff $\vec{v}_n \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\}$.

<u>Proof</u> 1) Suppose $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\} = \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$. Then since $\vec{v}_n \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ (it is $0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_{n-1} + 1 \cdot \vec{v}_n$), $\vec{v}_n \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\}$ as well.

2) Suppose that $\vec{v}_n \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\}$; say that

$$\vec{v}_n = d_1 \vec{v}_1 + \cdots + d_{n-1} \vec{v}_{n-1}.$$

Then $\forall \vec{v} \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$,

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n \quad \text{for some } c_1, \cdots, c_n \in \mathbb{R},$$

$$\Rightarrow \quad \vec{v} = c_1 \vec{v}_1 + \cdots + c_{n-1} \vec{v}_{n-1} + c_n \cdot (d_1 \vec{v}_1 + \cdots + d_{n-1} \vec{v}_{n-1})$$

$$= (c_1 + d_1 \cdot c_n) \vec{v}_1 + (c_2 + d_2 \cdot c_n) \vec{v}_2 + \cdots + (c_{n-1} + d_{n-1} \cdot c_n) \vec{v}_{n-1}$$

$\Rightarrow \quad \vec{v} \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\}$ as well.

So $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\}$. But also $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\} \subseteq \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ since $c_1 \vec{v}_1 + \cdots + c_{n-1} \vec{v}_{n-1} = c_1 \vec{v}_1 + \cdots + c_{n-1} \vec{v}_{n-1} + 0 \cdot \vec{v}_n$.

So $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\} = \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$, as desired.