math 272
10/18/17

## Review of some key concepts on vector spaces

- A **vector space** is a set of objects that can be scaled (by real numbers) and added together, in a way that respects a few algebraic laws (axioms).

  $\longrightarrow$ key examples: $\mathbb{R}^n$, $M_{m\times n}$ (m×n matrices), $P_d$ (polynomials of degree ≤d).

- A **subspace** is a subset that's closed under addition & scaling.

  e.g. subspaces of $\mathbb{R}^3$ are:

  $\longrightarrow$ all of $\mathbb{R}^3$ (3-dim'l)

  $\longrightarrow$ planes through the origin (2-dim'l)

  $\longrightarrow$ lines through the origin (1-dim'l)

  $\longrightarrow$ the set $\{\vec{0}\}$ (0-dim'l)

  □ Why do subspaces always contain $\vec{0}$?

- The **span** of a list $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in V$ is the set of all linear combinations $c_1 \odot \vec{v}_1 \oplus c_2 \odot \vec{v}_2 \oplus \cdots \oplus c_n \odot \vec{v}_n$.

  It is a subspace.

- A list $\vec{v}_1, \cdots, \vec{v}_n$ is **linearly independent** if linear combinations $\vec{v} \in \text{span}(\vec{v}_1, \cdots, \vec{v}_n)$ uniquely determine the coefficients $c_i$ s.t.

$$\vec{v} = c_1 \odot \vec{v}_1 \oplus \cdots \oplus c_n \odot \vec{v}_n.$$

  □ Convince yourself that this is equivalent to the usual defn. which states that $c_1 \odot \vec{v}_1 \oplus \cdots \oplus c_n \odot \vec{v}_n = \vec{0}$ holds only when $c_1 = c_2 = \cdots = c_n = 0$.

- A **basis** is an ordered list $\vec{v}_1, \cdots, \vec{v}_n \in V$ that is both linearly independent and spans all of $V$.

  $\longrightarrow$ This means that you can assign unique coordinates to each $\vec{v} \in V$.

  □ Convince yourself why this is true.

## Some review questions

Try to convince yourself on an intuitive level why these things are true. Then think about how you'd write a proof.

- ☐ A list $\vec{v}_1, \dots, \vec{v}_n$ is linearly independent iff no vector in the list is in the span of the others.

- ☐ If $\vec{v}_1, \dots, \vec{v}_n$ is linearly <u>dependent</u>, then every $\vec{v} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ can be written as a linear combination
$$\vec{v} = c_1 \odot \vec{v}_1 \oplus \cdots \oplus c_n \odot \vec{v}_n$$
in <u>infinitely</u> <u>many</u> ways.

- ☐ $\text{span}(\{\vec{v}_1, \dots, \vec{v}_{n-1}\}) = \text{span}(\{\vec{v}_1, \dots, \vec{v}_{n-1}, \vec{v}_n\})$
iff $\vec{v}_n \in \text{span}(\{\vec{v}_1, \dots, \vec{v}_{n-1}\})$.
(both statements express a sort of "redundancy" of $\vec{v}_n$).