Some remarks on "left-inverses" and "right-inverses."

// not essential to the course, but may help you put things in context.

Recall that a matrix B is an inverse of a matrix A if both

$$A \cdot B = I \qquad \text{and} \qquad B \cdot A = I \qquad \text{hold.}$$

I've told you (without proof, for now) that for square ($n \times n$) matrices, either $A \cdot B = I$ or $B \cdot A = I$ automatically implies the other. This handout shows how things can be different for non-square matrices.

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example 1 Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then A has a "right-inverse," namely $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, since $A \cdot B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the 2×2 identity.

This means in part that solutions exist to any linear system $A\vec{x} = \vec{b}$:

if $\vec{x} = B \cdot \vec{b}$, then $A\vec{x} = (AB)\vec{b} = \vec{b}$

so $\vec{x} = B \cdot \vec{b}$ is a solution to $A\vec{x} = \vec{b}$.

eg. one solution to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}$.

But the solution isn't unique: $\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$ is another.

In fact, the right-inverse itself isn't unique; $B' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & -1 \end{pmatrix}$ is also a "right-inverse."

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example 2 Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 6 \end{pmatrix}$. Then A has a "left-inverse," $B = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \end{pmatrix}$, since $\overbrace{\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \end{pmatrix}}^{B} \overbrace{\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 6 \end{pmatrix}}^{A} = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{I}$.

This means that if a solution to a linear system exists, then it is necessarily unique.

$$\text{if} \quad A\vec{x} = \vec{b}, \text{ then } \quad \vec{x} = (BA)x = B\vec{b}.$$

ie. $\vec{x} = B\vec{b}$ is the only possible solution.

eg. $\vec{x} = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the unique sol'n

to $\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 6 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 5 \\ 11 \end{pmatrix}$.

But the solution doesn't necessarily exist.. eg. $\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ would be the only possible sol'n to $A\vec{x} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$... but it isn't actually a solution!

So: B can tell you a candidate solution; if it is actually a sol'n, then it's the unique one, but if it isn't then there isn't a solution at all.

Also, the left-inverse isn't unique either, eg. $B' = \begin{pmatrix} 10 & -8 & 1 \\ 5 & -5 & 1 \end{pmatrix}$ is another one.

This is part of what makes square matrices so special. There is no need to worry about left versus right inverses, and existence & uniqueness of solution to linear systems go hand in hand.