Supplement to problem set 10
(Concerns material from §5.2 and §5.4)

You may use Mathematica (or any other program) to perform row-reduction or invert matrices in solving these problems; just write down the result of these computations and note that you used a computer for them. This will likely save some time and avoid error.

1. Let T be the transition matrix from problem 5.4.4, i.e.

$$T = \begin{pmatrix} 0.6 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

   (a) Diagonalize the matrix T.

   (b) Find an explicit formula for T^n (in the form shown in class). Use your formula to check your answers to part (b) of 5.4.4.

   (c) Determine $\lim_{n\to\infty} T^n$, as a matrix.

2. The *Fibonacci numbers* are a sequence $F_0, F_1, F_2, \cdots$ defined as follows: $F_0 = 0,\ F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The sequence begins $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$. In this problem, you will use diagonalization to find an explicit formula, called "Binet's formula," for the nth Fibonacci number.

   (a) Show that for all $n \geq 1$,

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}.$$

   Deduce from this that,

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   (b) Diagonalize the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and use this to find an explicit formula for the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$.

   (c) Combine parts (a) and (b) to obtain an explicit formula for the number F_n.

3. Define a new sequence $G_0, G_1, G_2, \cdots$ as follows: $G_0 = 0,\ G_1 = 1$, and $G_n = G_{n-1} + 2G_{n-2}$ for all $n \geq 2$. The sequence begins $0, 1, 1, 3, 5, 11, 21, \cdots$. Following the same method as the previous problem, find an explicit formula for the number G_n in terms of n.