Suggested reading for this week (from the textbook): §6.4 (ordering cardinalities)
Problems from the book: (First two numbers refer to the section number)

- 6.2.3 (Finding inverse functions when they exist; nine parts)

Study items for PSet 9:

- What it means for two functions to be equal.
- Relationship between function composition and injective/surjective/bijective.
- Inverse functions in terms of graphs (and examples).
- Inverse functions, definition in terms of function composition.
- Definition: the identity functions id_S on a set S.
- Know how to prove: a function has an inverse iff it is bijective.
- Uniqueness of inverse functions.

Supplemental problems:

1. This problem uses some techniques from calculus. You may assume the following two facts without proof (you practice rigorously proving facts like this in Analysis, e.g. Math 355).

   - If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} = \infty$, then f is surjective.
   - If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.

   For each of the following functions, determine whether or not the function has an inverse function, and prove your answer. *You do not need to find a formula for the inverse function, you need only prove that it exists (or that it does not).*

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^5 + x^3 + x$.
   (b) $f : \mathbb{R} \to \mathbb{R}$ defiend by $f(x) = x^3 - x$.
   (c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \arctan x$.
   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x - 3e^{-x}$.

2. Suppose that $f : A \to B$ and $g : B \to A$ are functions such that $g \circ f = \mathrm{id}_A$ and g is injective. Prove that f and g are inverse functions. (This was mentioned in the pre-recorded video from 4/8; it is closely related to Theorem 3 from that video).

3. Consider the function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^4$.

   (a) Prove that f does not have an inverse.
   (b) Find a function $g : [0, \infty) \to \mathbb{R}$ such that $f \circ g = \mathrm{id}_{[0,\infty)}$. Explain why this does not contradict part (a).

4. In this problem and the ones after it, we make the following definition: if A, B are sets, and $f : A \to B$ is a function, then

- A *left-inverse* of f is a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.
- A *right-inverse* of f is a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.

(a) Prove that if f has a left-inverse, then f is injective.

(b) Prove that if f has a right-inverse, then f is surjective.

5. Prove that if f has *both* a left-inverse g_ℓ and a right-inverse g_r, then $g_\ell = g_r$ and f is invertible.

6. (a) Give an example of a function $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ that has a left-inverse but not a right-inverse (explicitly describe the left-inverse, and prove that there is no right-inverse).

   (b) Give an example of a function $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$ that has a right-inverse but not a left-inverse (explicitly describe the right-inverse, and prove that there is no left-inverse).