Note: These problems are meant for you to practice with the last week of material, but you are NOT required to write up solutions or turn anything in. I will post instead solutions on the website so that you can check your work.

Suggested reading for this week (from the textbook): §8.3, §8.4

Problems from the book: (First two numbers refer to the section number)

• 8.3.2 \( \lim_{n \to \infty} \frac{2n - 3}{4 - 5n} \)

Solution: Observe that we can rewrite the “absolute error” as follows.

\[
\left| \frac{2n - 3}{4 - 5n} - \left( -\frac{2}{5} \right) \right| = \left| \frac{5(2n - 3) + 2(4 - 5n)}{(4 - 5n) \cdot 5} \right| = \left| \frac{10n - 15 + 8 - 10n}{20 - 25n} \right| = \left| \frac{7}{25n - 20} \right|
\]

Let \( \varepsilon \) be any positive number. Then we may define \( N \) to be any integer greater than \( \frac{1}{25} \left( \frac{7}{\varepsilon} + 20 \right) \) (e.g. round this expression up to an integer and add one). It follow that for all \( n \geq N \), we have \( 25n - 20 > \frac{7}{\varepsilon} \), and therefore \( \left| \frac{7}{25n - 20} \right| < \varepsilon \). This shows that for all \( \varepsilon \in (0, \infty) \), there exists \( N \) such that for all \( n \geq N \) we have \( \left| \frac{2n - 3}{4 - 5n} - \left( -\frac{2}{5} \right) \right| < \varepsilon \); this is what it means to say that \( \lim_{n \to \infty} \frac{2n - 3}{4 - 5n} = -\frac{2}{5} \).

• 8.3.3 \( \lim_{n \to \infty} \frac{1 - n^2}{3n^2 + 1} \)

Solution: (In order to show a different way to organize the work, I present this in a different order than the previous problem. Note that you’d almost certainly want to do some scratch-work where you “work backwards” in order to write this proof “forwards.”)

Let \( \varepsilon \) be any positive number. Let \( N = \left\lceil \frac{1}{2\sqrt{\varepsilon}} \right\rceil \) (that is, round \( \frac{1}{2\sqrt{\varepsilon}} \) up to an integer). Now observe that for all \( n \geq N \),

\[
3n^2 \geq \frac{3}{4\varepsilon} \\
\Rightarrow 3n^2 + 1 > \frac{3}{4\varepsilon} \\
\Rightarrow \frac{4}{3n^2 + 1} < \varepsilon \\
\Rightarrow \left| \frac{4}{3n^2 + 1} \right| < \varepsilon
\]

Finally, observe that \( \frac{1 - n^2}{3n^2 + 1} + \frac{1}{3} = \frac{4/3}{3n^2 + 1} \), so the above inequality is equivalent to

\[
\left| \frac{1 - n^2}{3n^2 + 1} - \left( -\frac{1}{3} \right) \right| < \varepsilon.
\]
This shows that \( \lim_{n \to \infty} \frac{1 - n^2}{3n^2 + 1} = -\frac{1}{3} \), as desired.

- **8.3.6 (convergent sequences of integers are eventually constant)**

  Solution: Let \((z_n)_{n=1}^{\infty}\) be a convergent sequence of integers, and suppose that \( \lim_{n \to \infty} z_n = L \). Let \( \varepsilon = \frac{1}{2} \) in the definition of “limit;” it follows that there exists a positive integer \( N \) such that for all \( n \geq N \) we have \( |z_n - L| < \frac{1}{2} \). From this and the triangle inequality, it follows that for all \( n \neq N \), we have

  \[
  |z_n - z_N| = |(z_n - L) - (z_N - L)| \leq |z_n - L| + |z_N - L| < \frac{1}{2} + \frac{1}{2} = 1.
  \]

  Now, \( |z_n - z_N| < 1 \) implies that \( z_n = z_N \), since any two distinct integers must differ by at least 1. Therefore this demonstrates that for this value of \( N \), we have \( a_n = a_N \) for all \( n \geq N \), as desired.

- **9.3.12(a) (shifting a sequence does not change the limit)**

  Solution: First, suppose that \( \lim_{n \to \infty} a_n = L \). Let \( \varepsilon \) be any real number. By definition of “limit,” there exists a natural number \( N \) such that for all \( n \geq N \), we have \( |a_n - L| < \varepsilon \). For this same value of \( N \), it follows that for all \( n \geq N \), we have \( n + 100 \geq N \) and thus \( |a_{n+100} - L| < \varepsilon \), i.e. \( |b_n - L| < \varepsilon \). Therefore we also have \( \lim_{n \to \infty} b_n = L \).

  Conversely, suppose that \((b_n)_{n=1}^{\infty}\) converges to \( L \). Let \( \varepsilon \) be any positive real number. Then there exists a natural number \( N_0 \) such that for all \( n \geq N_0 \), \( |b_n - L| < \varepsilon \). Let \( N = N_0 + 100 \). Then it follows that for all \( n \geq N \), we have \( n - 100 \geq N \), and hence \( |b_{n-100} - L| < \varepsilon \), i.e. \( |a_n - L| < \varepsilon \). This shows that for all \( \varepsilon > 0 \), there exists \( N \) such that for all \( n \geq N \) we have \( |a_n - L| < \varepsilon \). Therefore we also have \( \lim_{n \to \infty} a_n = L \), as desired.

**Supplemental problems:**

1. Suppose that \((a_n)_{n=1}^{\infty}\) is a convergent sequence of positive real numbers, with limit \( L \). Prove that \((\sqrt{a_n})_{n=1}^{\infty}\) is also convergent, and has limit \( \sqrt{L} \).

   Solution:

   We can do a bit of algebra on the expression for the error, using the “rationalize the denominator” algebraic trick from calculus:

   \[
   |\sqrt{a_n} - \sqrt{L}| = \left| \frac{\sqrt{a_n} - \sqrt{L}}{1} \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} \right|
   = \left| \frac{a_n - L}{\sqrt{a_n} + \sqrt{L}} \right|
   \]

   Since \( \sqrt{a_n} \) is positive, we could bound this error as follows:

   \[
   |\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} = \frac{|a_n - L|}{\sqrt{L}}.
   \]

   (This is a somewhat weak bound, but it won’t matter for purposes of this proof.)
Now we’re in a good positive to choose our “threshold” $N$. Let $\varepsilon$ be any positive real number. Since $\lim_{n \to \infty} a_n = L$, there exists a natural number $N$ such that for all $n \geq N$, we have

$$|a_n - L| < \sqrt{L} \cdot \varepsilon.$$ 

It follows from this that for all $n \geq N$,

$$\left| \sqrt{a_n} - \sqrt{L} \right| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\sqrt{L} \cdot \varepsilon}{\sqrt{L}} = \varepsilon.$$ 

So indeed it follows that $\lim_{n \to \infty} \sqrt{a_n} = \sqrt{L}$, as desired.

2. Prove the “decreasing” version of the monotone convergence theorem: if $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of real numbers, then $(a_n)_{n=1}^{\infty}$ converges if and only if it is bounded below.

There are at least two ways to go about this.

Solution 1 (prove using the “increasing” version):

Consider the sequence $(-a_n)_{n=1}^{\infty}$. This is an increasing sequence, since for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$ implies that $-a_{n+1} \geq -a_n$. By the “increasing” version of the monotone convergence theorem, $(-a_n)_{n=1}^{\infty}$ converges. By one of the limit laws proved in class (the one concerning multiplying a sequence by a scalar), the sequence $((-1) \cdot (a_n))_{n=1}^{\infty}$ also converges, i.e. $(a_n)_{n=1}^{\infty}$ converges, as desired.

Solution 2 (mimicing the proof of the increasing version):

First, we prove that the completeness axiom for $\mathbb{R}$ also implies that any set of real numbers that is bounded below must have a greatest lower bound. This can be proved from the completeness axiom: if $S$ is bounded below, with lower bound $B$, then $T = \{ -x : x \in S \}$ is bounded above by $-B$ since $-x \leq -B$ for all $x \in S$. Hence by the completeness axiom, $T$ has a least upper bound $L$. It follows that for all $x \in S$, $-x \leq L$, hence $x \geq -L$; so $-L$ is a lower bound for $S$. Also, if $L'$ is any real number less than $L$, then $L'$ is not an upper bound for $T$; i.e. there exists $x \in S$ such that $-x > L'$. It follows that $-L'$ is not a lower bound for $S$. This shows that any real number greater than $-L$ is not a lower bound for $S$, so $-L$ is indeed a greatest lower bound.

Now, we use the existence of greatest lower bounds to prove the desired result. Let $G$ denote the greatest lower bound of the set $\{a_n : n \in \mathbb{N}\}$. This number exists since we have assumed that the sequence is bounded below.

For all positive real numbers $\varepsilon$, $G + \varepsilon$ is not a lower bound for the sequence (since $G$ is the greatest lower bound), so there exists $N \in \mathbb{N}$ such that $a_N < G + \varepsilon$. Since the sequence is decreasing, it follows that for all $n \geq N$, $a_n \leq a_N < G + \varepsilon$. In other words, $a_n - G < \varepsilon$. Since $G$ is a lower bound for $a_n$, we have $a_n - G \geq 0$, so $|a_n - G| = a_n - G < \varepsilon$. This demonstrates that the sequence $(a_n)_{n=1}^{\infty}$ converges to $G$, its greatest lower bound.

3. (The comparison test for infinite series) Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are two sequences of positive real numbers, such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Prove that if the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. (Hint: use the monotone convergence theorem.)

Solution:
Let \( s_n = \sum_{k=1}^{n} a_n \) and let \( t_n = \sum_{k=1}^{n} b_n \). So \( (s_n)_{n=1}^{\infty} \) and \( (t_n)_{n=1}^{\infty} \) are the sequences of partial sums for the sequences \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \), respectively.

Since \( a_n \) is positive for all \( n \), it follows that \( s_{n+1} = s_n + a_{n+1} > s_n \) for all \( n \in \mathbb{N} \), and thus the sequence \( (s_n)_{n=1}^{\infty} \) is an increasing sequence. The same reasoning shows that \( (t_n)_{n=1}^{\infty} \) is an increasing sequence. So we can apply the monotone convergence theorem to both sequences.

We are assuming that the series \( \sum_{n=1}^{\infty} b_n \) converges. In other words, the sequence \( (t_n)_{n=1}^{\infty} \) converges. By the monotone convergence theorem, it is bounded above (in fact, it is bounded above by its limit). Let \( B \) be an upper bound. Then \( t_n \leq B \) for all \( n \in \mathbb{N} \).

Since \( a_n \leq b_n \) for all \( n \), it follows that \( s_n \leq t_n \) for all \( n \) (I will omit the formal proof of this for brevity, but try to write it out! It is a nice exercise in induction). Therefore \( s_n \leq t_n \leq B \) for all \( n \in \mathbb{N} \), and thus \( B \) is an upper bound for the sequence \( (s_n)_{n=1}^{\infty} \) as well. So \( (s_n)_{n=1}^{\infty} \) is an increasing sequence that is bounded above; by the monotone convergence theorem it also converges. By definition this means that the series \( \sum_{n=1}^{\infty} a_n \) converges, as desired.