Note: These problems are meant for you to practice with the last week of material, but you are NOT required to write up solutions or turn anything in. I will post instead solutions on the website so that you can check your work.

Suggested reading for this week (from the textbook): §8.3, §8.4

Problems from the book: (First two numbers refer to the section number)

- 8.3.2 $\left(\lim_{n\to\infty} \frac{2n-3}{4-5n}\right)$

  Solution: Observe that we can rewrite the "absolute error" as follows.

$$\begin{aligned}\left|\frac{2n-3}{4-5n} - \left(-\frac{2}{5}\right)\right| &= \left|\frac{5(2n-3)+2(4-5n)}{(4-5n)\cdot 5}\right| \\ &= \left|\frac{10n-15+8-10n}{20-25n}\right| \\ &= \left|\frac{7}{25n-20}\right|\end{aligned}$$

  Let ε be any positive number. Then we may define N to be any integer greater than $\frac{1}{25}\left(\frac{7}{\varepsilon}+20\right)$ (e.g. round this expression up to an integer and add one). It follow that for all $n \geq N$, we have $25n-20 > \frac{7}{\varepsilon}$, and therefore $\left|\frac{7}{25n-20}\right| < \varepsilon$. This shows that for all $\varepsilon \in (0,\infty)$, there exists N such that for all $n \geq N$ we have $\left|\frac{2n-3}{4-5n} - \left(-\frac{2}{5}\right)\right| < \varepsilon$; this is what it means to say that $\lim_{n\to\infty} \frac{2n-3}{4-5n} = -\frac{2}{5}$.

- 8.3.3 $\left(\lim_{n\to\infty} \frac{1-n^2}{3n^2+1}\right)$

  Solution: (In order to show a different way to organize the work, I present this in a different order than the previous problem. Note that you'd almost certainly want to do some scratch-work where you "work backwards" in order to write this proof "forwards.")

  Let ε be any positive number. Let $N = \lceil \frac{1}{2\sqrt{\varepsilon}} \rceil$ (that is, round $\frac{1}{2\sqrt{\varepsilon}}$ up to an integer). Now observe that for all $n \geq N$,

$$\begin{aligned} 3n^2 &\geq \frac{3}{4\varepsilon} \\ \Rightarrow 3n^2+1 &> \frac{3}{4\varepsilon} \\ \Rightarrow \frac{4/3}{3n^2+1} &< \varepsilon \\ \Rightarrow \left|\frac{4/3}{3n^2+1}\right| &< \varepsilon \end{aligned}$$

  Finally, observe that $\frac{1-n^2}{3n^2+1} + \frac{1}{3} = \frac{4/3}{3n^2+1}$, so the above inequality is equivalent to

$$\left|\frac{1-n^2}{3n^2+1} - \left(-\frac{1}{3}\right)\right| < \varepsilon.$$

This shows that $\lim_{n\to\infty} \frac{1-n^2}{3n^2+1} = -\frac{1}{3}$, as desired.

- 8.3.6 (convergent sequences of integers are eventually constant)

  Solution: Let $(z_n)_{n=1}^{\infty}$ be a convergent sequence of integers, and suppose that $\lim_{n\to\infty} z_n = L$. Let $\varepsilon = \frac{1}{2}$ in the definition of "limit;" it follows that there exists a positive integer N such that for all $n \geq N$ we have $|z_n - L| < \frac{1}{2}$. From this and the triangle inequality, it follows that for all $n \neq N$, we have

  $$|z_n - z_N| = |(z_n - L) - (z_N - L)| \leq |z_n - L| + |z_N - L| < \frac{1}{2} + \frac{1}{2} = 1.$$

  Now, $|z_n - z_N| < 1$ implies that $z_n = z_N$, since any two distinct integers must differ by at least 1. Therefore this demonstrates that for this value of N, we have $a_n = a_N$ for all $n \geq N$, as desired.

- 9.3.12(a) (shifting a sequence does not change the limit)

  Solution: First, suppose that $\lim_{n\to\infty} a_n = L$. Let ε be any real number. By definition of "limit," there exists a natural number N such that for all $n \geq N$, we have $|a_n - L| < \varepsilon$. For this same value of N, it follows that for all $n \geq N$, we have $n + 100 \geq N$ and thus $|a_{n+100} - L| < \varepsilon$, i.e. $|b_n - L| < \varepsilon$. Therefore we also have $\lim_{n\to\infty} b_n = L$.

  Conversely, suppose that $(b_n)_{n=1}^{\infty}$ converges to L. Let ε be any positive real number. Then there exists a natural number N_0 such that for all $n \geq N_0$, $|b_n - L| < \varepsilon$. Let $N = N_0 + 100$. Then it follows that for all $n \geq N$, we have $n - 100 \geq N$, and hence $|b_{n-100} - L| < \varepsilon$, i.e. $|a_n - L| < \varepsilon$. This shows that for all $\varepsilon > 0$, there exists N such that for all $n \geq N$ we have $|a_n - L| < \varepsilon$. Therefore we also have $lim_{n\to\infty} a_n = L$, as desired.

Supplemental problems:

1. Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence of positive real numbers, with limit L. Prove that $(\sqrt{a_n})_{n=1}^{\infty}$ is also convergent, and has limit $\sqrt{L}$. Solution:

   We can do a bit of algebra on the expression for the error, using the "rationalize the denominator" algebraic trick from calculus:

   $$\begin{aligned} \left|\sqrt{a_n} - \sqrt{L}\right| &= \left|\frac{\sqrt{a_n} - \sqrt{L}}{1} \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}}\right| \\ &= \left|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\right| \end{aligned}$$

   Since $\sqrt{a_n}$ is positive, we could bound this error as follows:

   $$\left|\sqrt{a_n} - \sqrt{L}\right| < \left|\frac{a_n - L}{\sqrt{L}}\right| = \frac{|a_n - L|}{\sqrt{L}}.$$

   (This is a somewhat weak bound, but it won't matter for purposes of this proof.)

Now we're in a good positive to choose our "threshold" N. Let ε be any positive real number. Since $\lim_{n\to\infty} a_n = L$, there exists a natural number N such that for all $n \geq N$, we have

$$|a_n - L| < \sqrt{L} \cdot \varepsilon.$$

It follows from this that for all $n \geq N$,

$$\left|\sqrt{a_n} - \sqrt{L}\right| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\sqrt{L} \cdot \varepsilon}{\sqrt{L}} = \varepsilon.$$

So indeed it follows that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$, as desired.

2. Prove the "decreasing" version of the monotone convergence theorem: if $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of real numbers, then $(a_n)_{n=1}^{\infty}$ converges if and only if it is bounded below.

   There are at least two ways to go about this.

   Solution 1 (prove using the "increasing" version):

   Consider the sequence $(-a_n)_{n=1}^{\infty}$. This is a an increasing sequence, since for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$ implies that $-a_{n+1} \geq -a_n$. By the "increasing" version of the monotone convergence theorem, $(-a_n)_{n=1}^{\infty}$ converges. By one of the limit laws proved in class (the one concerning multiplying a sequence by a scalar), the sequence $((-1) \cdot (-a_n))_{n=1}^{\infty}$ also converges, i.e. $(a_n)_{n=1}^{\infty}$ converges, as desired.

   Solution 2 (mimicing the proof of the increasing version):

   First, we prove that the completeness axiom for $\mathbb{R}$ also imples that any set of real numbers that is bounded *below* must have a *greatest lower bound*. This can be proved from the completness axiom: if S is bounded below, with lower bound B, then $T = \{-x : x \in S\}$ is bounded above by $-B$ since $-x \leq -B$ for all $x \in S$. Hence by the completeness axiom, T has a *least upper bound* L. It follows that for all $x \in S$, $-x \leq L$, hence $x \geq -L$; so $-L$ is a lower bound for S. Also, if L' is any real number less than L, then L' is *not* an upper bound for T, i.e. there exists $x \in S$ such that $-x > L'$. It follows that $-L'$ is not a lower bound for S. This shows that any real number greater than $-L$ is not a lower bound for S, so $-L$ is indeed a *greatest* lower bound.

   Now, we use the existence of greatest lower bounds to prove the desired result. Let G denote the greatest lower bound of the set $\{a_n : n \in \mathbb{N}\}$. This number exists since we have assumed that the sequence is bounded below.

   For all positive real numbers ε, $G + \varepsilon$ is not a lower bound for the sequence (since G is the *greatest* lower bound), so there exists $N \in \mathbb{N}$ such that $a_N < G + \varepsilon$. Since the sequence is decreasing, it follows that for all $n \geq N$, $a_n \leq a_N < G + \varepsilon$. In other words, $a_n - G < \varepsilon$. Since G is a lower bound for a_n, we have $a_n - G \geq 0$, so $|a_n - G| = a_n - G < \varepsilon$. This demonstrates that the sequence $(a_n)_{n=1}^{\infty}$ converges to G, its greatest lower bound.

3. (The comparison test for infinite series) Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are two sequences of *positive* real numbers, such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Prove that if the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. (*Hint:* use the monotone convergence theorem.)

   Solution:

Let $s_n = \sum_{k=1}^{n} a_n$ and let $t_n = \sum_{k=1}^{n} b_n$. So $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are the sequences of *partial sums* for the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, respectively.

Since a_n is positive for all n, it follows that $s_{n+1} = s_n + a_{n+1} > s_n$ for all $n \in \mathbb{N}$, and thus the sequence $(s_n)_{n=1}^{\infty}$ is an *increasing* sequence. The same reasoning shows that $(t_n)_{n=1}^{\infty}$ is an increasing sequence. So we can apply the monotone convergence theorem to both sequences.

We are assuming that the series $\sum_{n=1}^{\infty} b_n$ converges. In other words, the sequence $(t_n)_{n=1}^{\infty}$ converges. By the monotone convergence theorem, it is bounded above (in fact, it is bounded above by its limit). Let B be an upper bound. Then $t_n \leq B$ for all $n \in \mathbb{N}$.

Since $a_n \leq b_n$ for all n, it follows that $s_n \leq t_n$ for all n (I will omit the formal proof of this for brevity, but try to write it out! It is a nice exercise in induction). Therefore $s_n \leq t_n \leq B$ for all $n \in \mathbb{N}$, and thus B is an upper bound for the sequence $(s_n)_{n=1}^{\infty}$ as well. So $(s_n)_{n=1}^{\infty}$ is an increasing sequence that is bounded above; by the monotone convergence theorem it also converges. By definition this means that the series $\sum_{n=1}^{\infty} a_n$ converges, as desired.