Math 121 Final Exam May 14, 2016

1. [18 Points] Evaluate each of the following **limits**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to\infty}\left(\arcsin\left(\frac{1}{x}\right)+e^{\frac{1}{x}}\right)^x = e^{\lim\limits_{x\to\infty}\ln\left(\left(\arcsin\left(\frac{1}{x}\right)+e^{\frac{1}{x}}\right)^x\right)}$$

$$= e^{\lim\limits_{x\to\infty} x\ln\left(\arcsin\left(\frac{1}{x}\right)+e^{\frac{1}{x}}\right)}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\ln\left(\arcsin\left(\frac{1}{x}\right)+e^{\frac{1}{x}}\right)}{\frac{1}{x}}}\ \ \frac{0}{0}$$

$$\overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{1}{\arcsin\left(\frac{1}{x}\right)+e^{\frac{1}{x}}}\left(\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right)+e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\frac{1}{\arcsin\left(\frac{1}{x}\right)+e^{\frac{1}{x}}}\left(\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}+e^{\frac{1}{x}}\right)}{1}} = e^{1+1} = \boxed{e^2}$$

(b) $$\lim_{x\to 0}\frac{xe^x-\arctan x}{\ln(1+3x)-3x}\ \left(\tfrac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{xe^x+e^x-\frac{1}{1+x^2}}{\frac{3}{1+3x}-3}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{xe^x+e^x+e^x+\frac{2x}{(1+x^2)^2}}{-\frac{9}{(1+3x)^2}} = \boxed{-\frac{2}{9}}$$

(c) Compute $\lim\limits_{x\to 0}\dfrac{xe^x-\arctan x}{\ln(1+3x)-3x}$ **again** using series.

$$\lim_{x\to 0}\frac{xe^x-\arctan x}{\ln(1+3x)-3x} = \lim_{x\to 0}\frac{x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right)}{\left(3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\dots\right)-3x}$$

$$= \lim_{x\to 0} \frac{x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots - x + \frac{x^3}{3} - \frac{x^5}{5} + \ldots}{-\frac{9x^2}{2} + \frac{27x^3}{3} - \ldots}$$

$$= \lim_{x\to 0} \frac{x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots + \frac{x^3}{3} - \frac{x^5}{5} + \ldots}{-\frac{9x^2}{2} + \frac{27x^3}{3} - \ldots}$$

$$= \lim_{x\to 0} \frac{x^2\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x}{3} - \frac{x^2}{5} + \ldots\right)}{x^2\left(-\frac{9}{2} + 9x - \ldots\right)} = \frac{1}{\left(-\frac{9}{2}\right)} = \boxed{-\frac{2}{9}} \quad \text{Match!}$$

2. [18 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{(4\tan^2\theta + 4)^2}\,2\sec^2\theta\,d\theta$

$$= \int \frac{1}{(4\sec^2\theta)^2}\,2\sec^2\theta\,d\theta = \int \frac{1}{16\sec^4\theta}\,2\sec^2\theta\,d\theta = \frac{1}{8}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

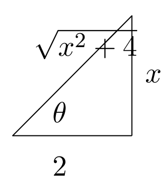
$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{16}\int 1 + \cos(2\theta)\,d\theta = \frac{1}{16}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= \frac{1}{16}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C = \frac{1}{16}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}}\left(\frac{2}{\sqrt{x^2+4}}\right)\right) + C = \boxed{\frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta d\theta \end{array}}$



(b) $\displaystyle\int_{-1}^{0} x^4 \arcsin x\,dx = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{-1}^{0} \frac{x^5}{\sqrt{1-x^2}}\,dx$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \frac{\sin^5\theta}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \frac{\sin^5\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0}\sin^5\theta\ d\theta = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0}\sin^4\theta\sin\theta\ d\theta$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0}(1-\cos^2\theta)^2\sin\theta\ d\theta = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\int_{x=-1}^{x=0}(1-w)^2\ dw$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\int_{x=-1}^{x=0}1-2w^2+w^4\ dw$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right)\Big|_{x=-1}^{x=0}$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\left(\cos\theta - \frac{2\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Big|_{x=-1}^{x=0}$$

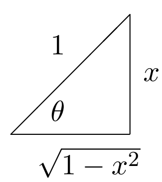
$$= \frac{x^5}{5}\arcsin x + \frac{\sqrt{1-x^2}}{5} - \frac{2(1-x^2)^{\frac{3}{2}}}{15} + \frac{(1-x^2)^{\frac{5}{2}}}{25}\Big|_{-1}^{0}$$

$$= 0 + \frac{1}{5} - \frac{2}{15} + \frac{1}{25} - \left(-\frac{1}{5}\arcsin(-1) + 0 - 0 + 0\right)$$

$$= \frac{1}{5} - \frac{2}{15} + \frac{1}{25} + \frac{1}{5}\left(-\frac{\pi}{2}\right) = \frac{15}{75} - \frac{10}{75} + \frac{3}{75} - \frac{\pi}{10} = \boxed{\frac{8}{75} - \frac{\pi}{10}}$$

$u = \arcsin x$	$dv = x^4 dx$
$du = \frac{1}{\sqrt{1-x^2}}dx$	$v = \frac{x^5}{5}$

Trig. Substitute $\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta d\theta \end{array}}$



Substitute $\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta d\theta \\ -dw = \sin\theta d\theta \end{array}}$

3. [36 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_0^1 \sqrt{x}\ \ln x\ dx = \lim_{t\to 0^+}\int_t^1 \sqrt{x}\ \ln x\ dx = \lim_{t\to 0^+}\ \frac{2}{3}x^{\frac{3}{2}}\ln x\Big|_t^1 - \frac{2}{3}\int_t^1 \sqrt{x}\ dx$

$$= \lim_{t\to 0^+}\ \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}}\Big|_t^1 = \lim_{t\to 0^+}\frac{2}{3}\ln 1 - \frac{2}{3}t^{\frac{3}{2}}\ln t - \left(\frac{4}{9} - \frac{4}{9}t^{\frac{3}{2}}\right)$$

$$\overset{(*)}{=} 0 - 0 - \frac{4}{9} + 0 = \boxed{-\frac{4}{9}} \text{ Converges}$$

$$(*) \quad \lim_{x\to 0^+} x^{\frac{3}{2}} \ln x^{0\cdot -\infty} = \lim_{x\to 0^+} \frac{\ln x}{x^{-\frac{3}{2}}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{3}{2x^{\frac{5}{2}}}} = \lim_{x\to 0^+} -\frac{2x^{\frac{3}{2}}}{3} = 0$$

Integration By Parts:

$u = \ln x$	$dv = \sqrt{x}\ dx$
$du = \frac{1}{x}dx$	$v = \frac{2}{3}x^{\frac{3}{2}}$

$$\text{(b)} \int_1^\infty \frac{e^{\frac{1}{x}}}{x^3}\ dx = \lim_{t\to\infty} \int_1^t \frac{e^{\frac{1}{x}}}{x^3}\ dx$$

$$= \lim_{t\to\infty} -\int_1^{\frac{1}{t}} ue^u\ du \overset{\text{IBP}}{=} \lim_{t\to\infty} -(ue^u - e^u)\Big|_1^{\frac{1}{t}} = \lim_{t\to\infty} -ue^u + e^u\Big|_1^{\frac{1}{t}}$$

$$= \lim_{t\to\infty} -\frac{1}{t}e^{\frac{1}{t}} + e^{\frac{1}{t}} - (-e + e) = 0 + 1 + 0 = \boxed{1} \quad \text{Converges}$$

Substitute

$u = \frac{1}{x}$
$du = -\frac{1}{x^2}dx$
$-du = \frac{1}{x^2}dx$

$x = 1 \Rightarrow u = \frac{1}{1} = 1$
$x = t \Rightarrow u = \frac{1}{t}$

Integration By Parts:

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

NOTE: $\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + C$

$$\text{(c)} \int_1^{\sqrt{3}} \frac{x^4 - x^3 + 3x^2 - x + 2}{x^3 - x^2 + 3x - 3}\ dx = \int_1^{\sqrt{3}} \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+3)}\ dx$$

$$= \lim_{t\to 1^+} \int_t^{\sqrt{3}} \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+3)}\ dx$$

$$= \lim_{t\to 1^+} \int_t^{\sqrt{3}} x + \frac{2x+2}{(x-1)(x^2+3)}\ dx$$

$$= \lim_{t\to 1^+} \int_t^{\sqrt{3}} x + \frac{1}{x-1} + \frac{-x+1}{x^2+3}\ dx$$

$$= \lim_{t\to 1^+} \int_t^{\sqrt{3}} x + \frac{1}{x-1} - \frac{x}{x^2+3} + \frac{1}{x^2+3}\ dx$$

$$= \lim_{t\to 1^+} \frac{x^2}{2} + \ln|x-1| - \frac{1}{2}\ln|x^2+3| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\Bigg|_t^{\sqrt{3}}$$

$$= \lim_{t\to 1^+} \frac{3}{2} + \ln|\sqrt{3}-1| - \frac{1}{2}\ln 6 + \frac{1}{\sqrt{3}}\arctan(1) - \left(\frac{t^2}{2} + \ln|t-1| - \frac{1}{2}\ln|t+3| + \frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)\right)$$

$$= \lim_{t\to 1^+} \frac{3}{2} + \ln|\sqrt{3}-1| - \frac{1}{2}\ln 6 + \frac{1}{\sqrt{3}}\left(\frac{\pi}{4}\right) - \left(\frac{1}{2} + \ln|t-1| - \frac{1}{2}\ln 4 + \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right)\right)$$

$= -(-\infty) + \text{all other terms finite} = \boxed{+\infty}$ Diverges

Long division yields:

$$x^3 - x^2 + 3x - 3\overline{)x^4 - x^3 + 3x^2 - x + 2}^{\;x}$$

$$\underline{-(x^4 - x^3 + 3x^2 - 3x)}$$

$$2x + 2$$

Partial Fractions Decomposition:

$$\frac{2x+2}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$$

Clearing the denominator yields:

$2x + 2 = A(x^2+3) + (Bx+C)(x-1)$
$2x + 2 = Ax^2 + 3A + Bx^2 - Bx + Cx - C$
$2x + 2 = (A+B)x^2 + (C-B)x + 3A - C$
so that $A + B = 0,\ C - B = 2$ and $3A - C = 2$
Solve for $A = 1,\ B = -1$ and $C = 1$

(d) $\displaystyle\int_{2\sqrt{3}}^{4} \frac{1}{\sqrt{16-x^2}}\ dx = \lim_{t\to 4^-} \int_{2\sqrt{3}}^{t} \frac{1}{\sqrt{16-x^2}}\ dx$

$$= \lim_{t\to 4^-} \arcsin\left(\frac{x}{4}\right)\Bigg|_{2\sqrt{3}}^{t} = \lim_{t\to 4^-} \arcsin\left(\frac{t}{4}\right) - \arcsin\left(\frac{2\sqrt{3}}{4}\right)$$

$$= \lim_{t\to 4^-} \arcsin\left(\frac{x}{4}\right)\Bigg|_{2\sqrt{3}}^{t} \lim_{t\to 4^-} \arcsin\left(\frac{t}{4}\right) - \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{2} - \frac{\pi}{3} = \boxed{\frac{\pi}{6}}$$ Converges

(e) $\displaystyle\int_7^\infty \frac{1}{x^2-8x+19}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{x^2-8x+19}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{(x-4)^2+3}\,dx$

$$= \lim_{t\to\infty}\int_3^{t-4} \frac{1}{w^2+3}\,dw = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{w}{\sqrt{3}}\right)\Bigg|_3^{t-4}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-4}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right)\right)$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-4}{\sqrt{3}}\right) - \arctan(\sqrt{3})\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$ Converges

Substitute $\boxed{\begin{array}{c} w = x-4 \\ dw = dx \end{array}}$ $\boxed{\begin{array}{c} x = 7 \Rightarrow w = 3 \\ x = t \Rightarrow w = t-4 \end{array}}$

4. [18 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,5^{n+2}}{2^{3n-1}} = -\frac{5^3}{2^2}+\frac{5^4}{2^5}-\frac{5^5}{2^8}+\ldots$

Here we have a geometric series with $a = -\dfrac{125}{4}$ and $r = -\dfrac{5}{2^3} = -\dfrac{5}{8}$

As a result, the sum is given by $\displaystyle\frac{a}{1-r} = \frac{-\frac{125}{4}}{1-\left(-\frac{5}{8}\right)} = \frac{-\frac{125}{8}}{\frac{13}{8}} = -\frac{125}{4}\cdot\frac{8}{13} = \boxed{-\frac{250}{13}}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,(\ln(27))^n}{3^{n+1}\,n!} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n\,(\ln(27))^n}{3^n\,n!} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{\left(-\frac{\ln 27}{3}\right)^n}{n!}$

$$= \frac{1}{3}e^{-\frac{\ln 27}{3}} = \frac{1}{3}e^{\ln\left(27^{-\frac{1}{3}}\right)} = \frac{1}{3}\left(\frac{1}{\sqrt[3]{27}}\right) = \frac{1}{3}\left(\frac{1}{3}\right) = \boxed{\frac{1}{9}}$$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{2^{4n}\,(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{4^{2n}\,(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$

(d) $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots = -\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\right) = -\ln(1+1) = \boxed{-\ln 2}$

(e) $-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots = (\arctan 1) - 1 = \boxed{\frac{\pi}{4}-1}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!} \cdot \frac{\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)} = \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= \frac{3}{\pi} \sin\left(\frac{\pi}{3}\right) = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{2\pi}}$$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (n^5+7)}{n^7+5}$

First examine the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+7}{n^7+5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

which is a convergent p-series with $p = 2 > 1$.

Next check

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^5+7}{n^7+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^7+7n^2}{n^7+5} = \lim_{n\to\infty} \frac{1+\frac{7}{n^5}}{1+\frac{5}{n^7}} = 1$ which is finite and non-zero ($0 < 1 < \infty$).

Therefore, these two series share the same behavior, and the absolute series is also convergent by Limit Comparison Test (LCT). (Note: the Original Series is Convergent by ACT.) Finally, we have $\boxed{\text{Absolute Convergence}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(\sqrt{3}\ n^2+1)}{n^2+\sqrt{3}}$

First examine the A.S. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(\sqrt{3}\ n^2+1)}{n^2+\sqrt{3}}$

Next bound the terms

$$\frac{\arctan(\sqrt{3}\ n^2+1)}{n^2+\sqrt{3}} < \frac{\frac{\pi}{2}}{n^2+\sqrt{3}} < \frac{\frac{\pi}{2}}{n^2}$$

and

$\displaystyle\frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a constant multiple of a convergent p-series with $p = 2 > 1$ and therefore convergent.

Finally, the absolute series is Convergent by CT, and therefore the orginal series is $\boxed{\text{A.C.}}$. (Note: The O.S. is convergent by ACT.)

(c) $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{\sqrt{3}\ n^2+1}{n^2+\sqrt{3}}\right)$

$\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$

since $\displaystyle\lim_{n\to\infty} \arctan\left(\frac{\sqrt{3}\ n^2+1}{n^2+\sqrt{3}}\right) = \arctan\left(\lim_{n\to\infty} \frac{(\sqrt{3}\ n^2+1)}{(n^2+\sqrt{3})} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}\right)$

$\displaystyle = \arctan\left(\lim_{n\to\infty} \frac{\sqrt{3}+\frac{1}{n^2}}{1+\frac{\sqrt{3}}{n^2}}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3} \neq 0$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ n}{n^2+1}$

First, we show the absolute series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2+1}$ is divergent using LCT.

$\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2+1} \approx \sum_{n=2}^{\infty} \frac{1}{n}$ which is the divergent Harmonic p-series with $p=1$.

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$ which is finite and non-zero.

Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then the absolute series also diverges by LCT.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $\displaystyle b_n = \frac{n}{n^2+1} > 0$ for $n \geq 2$

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0$

- $b_{n+1} < b_n$ since we can show the derivative of the related function is negative, hence the terms are decreasing

Consider $\displaystyle f(x) = \frac{x}{x^2+1}$ with $\displaystyle f'(x) = \frac{-x^2+1}{(x^2+1)^2} < 0$ for $x > 1$

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the O.S. is $\boxed{\text{Conditionally Convergent}}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (\ln n)\ (3n)!}{n^n\ 2^{4n}\ (n!)^2}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}\ln(n+1)\ (3(n+1))!}{(n+1)^{n+1}2^{4(n+1)}((n+1)!)^2}}{\dfrac{(-1)^n\ \ln n\ (3n)!}{n^n\ 2^{4n}(n!)^2}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{(3n+3)!}{(3n)!}\right)\left(\frac{n^n}{(n+1)^{n+1}}\right)\left(\frac{(n!)^2}{((n+1)!)^2}\right)\left(\frac{2^{4n}}{2^{4n+4}}\right)\left(\frac{\ln(n+1)}{\ln n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!}\right)\left(\frac{n^n}{(n+1)^n(n+1)}\right)\left(\frac{(n!)^2}{(n+1)^2(n!)^2}\right)\left(\frac{2^{4n}}{2^{4n}2^4}\right)\left(\frac{\ln(n+1)}{\ln n}\right)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \left(\frac{3(n+1)(3n+2)(3n+1)}{1}\right)\left(\frac{1}{e}\right)\left(\frac{1}{n+1}\right)\left(\frac{1}{(n+1)^2}\right)\left(\frac{1}{16}\right)(1)$$

$$= \lim_{n\to\infty} \left(\frac{3}{16e}\right)\left(\frac{3n+2}{n+1}\right)\left(\frac{3n+1}{n+1}\right) = \lim_{n\to\infty} \left(\frac{3}{16e}\right)\left(\frac{3+\frac{2}{n}}{1+\frac{1}{n}}\right)\left(\frac{3+\frac{1}{n}}{1+\frac{1}{n}}\right)$$

$$= \frac{27}{16e} < 1$$

Therefore the original series $\boxed{\text{Converges Absolutely by the Ratio test}}$.

Here, from above,

$$(*) = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\infty}{\underset{\infty}{}} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\infty}{\underset{\infty}{}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{1} = 1$$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ (3x-5)^n}{n^8 \cdot 7^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(3x-5)^{n+1}}{(n+1)^8 7^{n+1}}}{\dfrac{(-1)^n(3x-5)^n}{n^8 7^n}} \right|$$

$$= \lim_{n\to\infty} \left|\frac{(3x-5)^{n+1}}{(3x-5)^n}\right| \left(\frac{n}{n+1}\right)^8 \left(\frac{7^n}{7^{n+1}}\right) = \lim_{n\to\infty} |3x-5| \left(\frac{1}{1+\frac{1}{n}}\right)^8 \left(\frac{1}{7}\right) = \frac{|3x-5|}{7}$$

The Ratio Test gives convergence for x when $\dfrac{|3x-5|}{7} < 1$ or $|3x-5| < 7$.

That is $-7 < 3x - 5 < 7 \Longrightarrow -2 < 3x < 12 \Longrightarrow -\dfrac{2}{3} < x < 4$

Endpoints:

•$x = -\dfrac{2}{3}$ The original series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left(3\left(-\frac{2}{3}\right)-5\right)^n}{n^8 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \ (-7)^n}{n^8 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \ (-1)^n 7^n}{n^8 \ 7^n} = \sum_{n=1}^{\infty} \frac{1}{n^8}$$

which is a convergent p-series with $p = 8 > 1$.

•$x = 4$ The original series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ (3\,(4)-5)^n}{n^8 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \ 7^n}{n^8 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^8}$$

Consider the A.S. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8}$ which was shown to be convergent above. Therefore the alternating O.S. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8}$ is convergent by ACT. (Note: you could also use AST)

Finally, Interval of Convergence $\boxed{I = \left[-\dfrac{2}{3}, 4\right]}$ with Radius of Convergence $\boxed{R = \dfrac{7}{3}}$.

7. [20 Points] Consider the region bounded by $y = \arctan x$, $y = 0$, $x = 0$ and $x = 1$. Rotate the region about the y–axis.

(a) **Sketch** the resulting solid, along with one of the approximating cylindrical shells.

See me for a sketch.

(b) **Set-up** the integral to compute the volume of this solid using the Cylindrical Shells Method.

$$V = \int_0^1 2\pi \text{ radius height } dx = \boxed{2\pi \int_0^1 x \ \arctan x \ dx}$$

(c) **Compute** your integral in part (b) above.

$$V = 2\pi \int_0^1 x \ \arctan x \ dx \overset{(**)}{=} 2\pi \left[\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x\right]\Bigg|_0^1$$

$$= \pi \left[x^2 \arctan x - x + \arctan x\right]\Bigg|_0^1 = \pi\left[(\arctan 1 - 1 + \arctan 1) - (0 \arctan 0 - 0 + \arctan 0)\right]$$

$$= \pi\left[\left(\frac{\pi}{4} - 1 + \frac{\pi}{4}\right) - (0\arctan 0 - 0 + \arctan 0)\right] = \pi\left[\frac{\pi}{2} - 1\right] = \boxed{\frac{\pi^2}{2} - \pi}$$

$$(**)\int x\ \arctan x\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C = \boxed{\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C}$$

$u = \arctan x$	$dv = xdx$
$du = \dfrac{1}{1+x^2}dx$	$v = \dfrac{x^2}{2}$

OR if you don't like the "slip-in/slip out" technique, use a tangent trig. substitution instead to finish the second piece of the I.B.P. $\int \frac{x^2}{1+x^2}\ dx$ $\int \frac{x^2}{1+x^2}\ dx = \int \frac{\tan^2\theta}{1+\tan^2\theta}\sec^2\theta\ d\theta =$ $\int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\ d\theta = \int \tan^2\theta\ d\theta = \int \sec^2\theta - 1\ d\theta$

$= \tan\theta - \theta = x - \arctan x$

Trig. Substitute

$x = \tan\theta$
$dx = \sec^2\theta d\theta$

(d) Use MacLaurin Series to **Estimate** the integral in part (b) above with error less than $\frac{2\pi}{20}$. Justify.

$$2\pi\int_0^1 x\ \arctan x\ dx = 2\pi\int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\ dx = 2\pi\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2n+1}\ dx$$

$$= 2\pi\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+1)(2n+3)}\right)\Bigg|_0^1 = 2\pi\left(\frac{x^3}{1\cdot 3} - \frac{x^5}{3\cdot 5} + \frac{x^7}{5\cdot 7} - \ldots\right)\Bigg|_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \ldots - (0 - 0 + 0 - \ldots)\right) \approx \frac{2\pi}{3} - \frac{2\pi}{15} = \boxed{\frac{8\pi}{15}} \quad \leftarrow \quad \textbf{estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the two terms, and the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\frac{2\pi}{35}$. Here $\frac{2\pi}{35} < \frac{2\pi}{20}$ as desired.

8. [20 Points] Consider the Parametric Curve represented by $x = e^t + \frac{1}{1+e^t}$ and $y = 2\ln(1+e^t)$.

(a) Write the **equation of the tangent line** to this curve at the point where $t = 0$.

First $\frac{dx}{dt} = e^t - \frac{e^t}{(1+e^t)^2}$ and $\frac{dy}{dt} = \frac{2e^t}{1+e^t}$.

Slope:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dy}{dt}} = \frac{\frac{2e^t}{1+e^t}}{e^t - \frac{e^t}{(1+e^t)^2}}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\frac{2e^0}{1+e^0}}{e^0 - \frac{e^0}{(1+e^0)^2}} = \frac{\frac{2}{2}}{1 - \frac{1}{(2)^2}} = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$$

Point: $(x(0), y(0)) = \left(\frac{3}{2}, 2\ln 2\right)$

Equation of the Tangent Line:

$y - 2\ln 2 = \frac{4}{3}\left(x - \frac{3}{2}\right)$ OR $y = \boxed{\frac{4}{3}x - 2 + 2\ln 2}$.

(b) **COMPUTE** the **arclength** of this parametric curve for $0 \le t \le \ln 3$.

$$L = \int_0^{\ln 3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\ln 3} \sqrt{\left(e^t - \frac{e^t}{(1+e^t)^2}\right)^2 + \left(\frac{2e^t}{1+e^t}\right)^2}\, dt$$

$$= \int_0^{\ln 3} \sqrt{e^{2t} - \frac{2e^{2t}}{(1+e^t)^2} + \frac{e^{2t}}{(1+e^t)^4} + \frac{4e^{2t}}{(1+e^t)^2}}\, dt$$

$$= \int_0^{\ln 3} \sqrt{e^{2t} + \frac{2e^{2t}}{(1+e^t)^2} + \frac{e^{2t}}{(1+e^t)^4}}\, dt$$

$$= \int_0^{\ln 3} \sqrt{\left(e^t + \frac{e^t}{(1+e^t)^2}\right)^2}\, dt = \int_0^{\ln 3} e^t + \frac{e^t}{(1+e^t)^2}\, dt$$

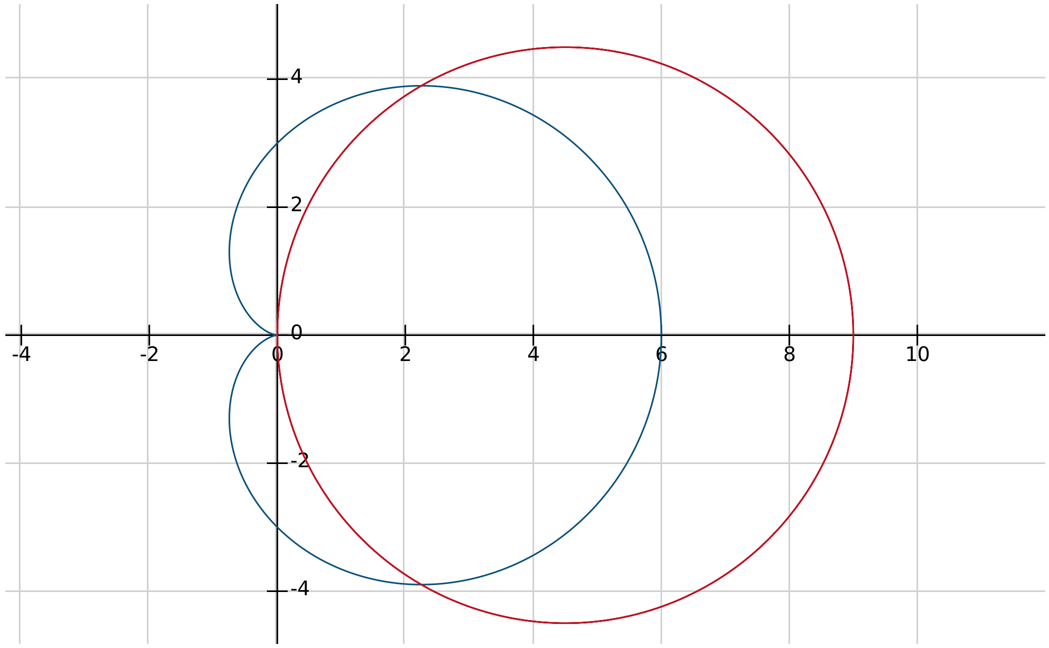
$$= e^t - \left.\frac{1}{1+e^t}\right|_0^{\ln 3} = e^{\ln 3} - \frac{1}{1+e^{\ln 3}} - \left(e^0 - \frac{1}{1+e^0}\right) = 3 - \frac{1}{4} - 1 + \frac{1}{2} = \boxed{\frac{9}{4}}$$

9. [20 Points] For each of the following parts, do the following **two** things:

1. Sketch the Polar curves and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(a) The **area** bounded outside the polar curve $r = 3 + 3\cos\theta$ and inside the polar curve



$r = 9\cos\theta$. OB

These two polar curves intersect when $3 + 3\cos\theta = 9\cos\theta \Rightarrow 6\cos\theta = 3 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\dfrac{\pi}{3}$.

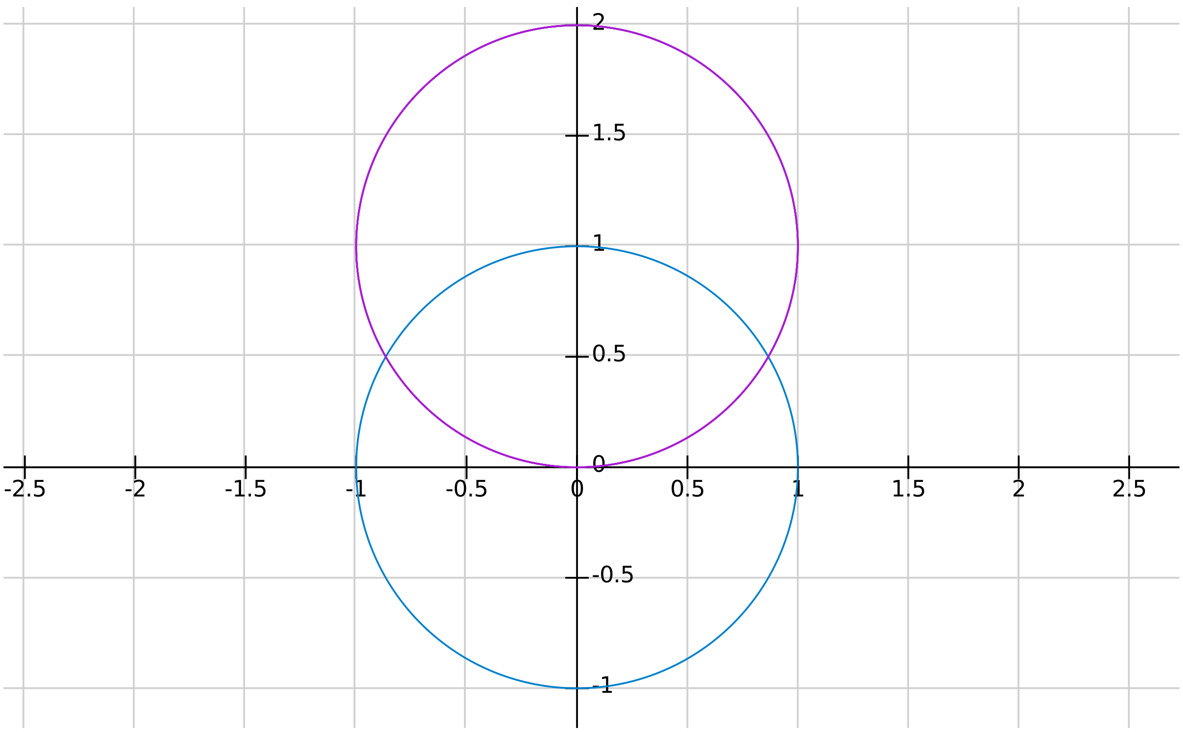
Area $= A = \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta$

$$= \boxed{\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((9\cos\theta)^2 - (3+3\cos\theta)^2\right)\ d\theta}$$

OR using symmetry

$$A = \boxed{2\left(\frac{1}{2}\int_{0}^{\frac{\pi}{3}} (9\cos\theta)^2 - (3+3\cos\theta)^2\ d\theta\right)}$$

(b) The **area** bounded outside the polar curve $r = 1$ and inside the polar curve $r = 2\sin\theta$.



These two polar curves intersect when

$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{5\pi}{6}$.

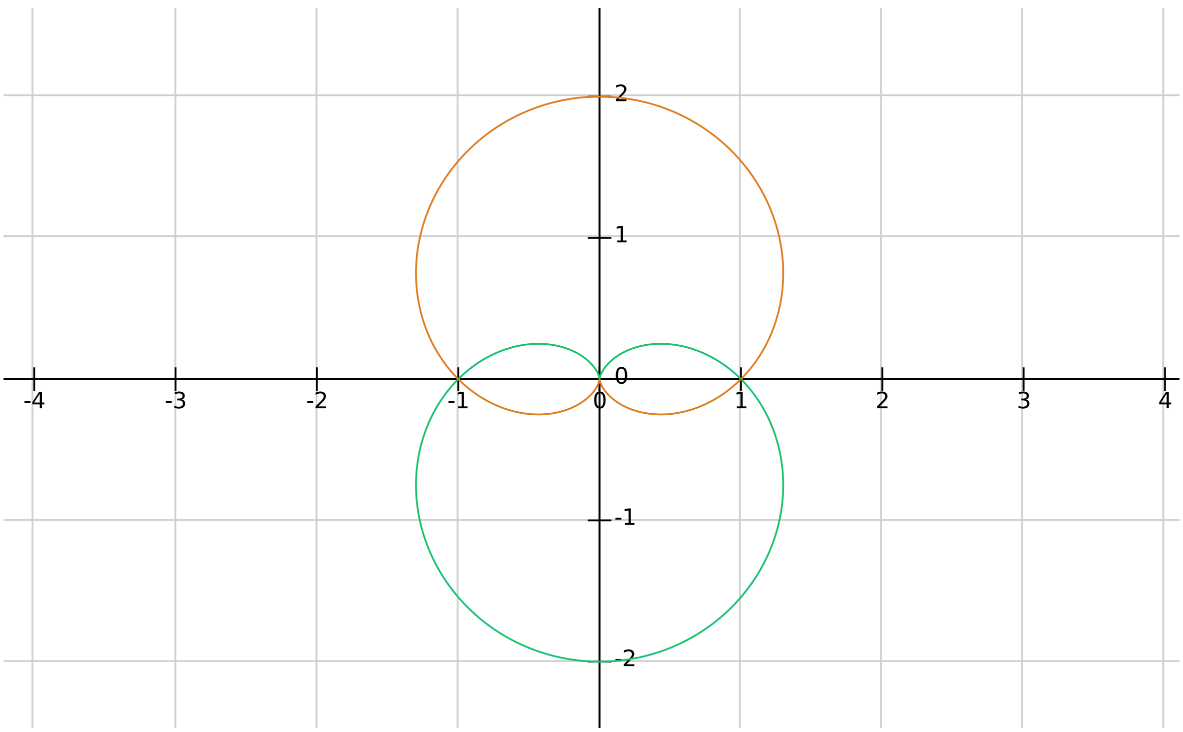
$$\text{Area} = A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((\text{outer} \ \ r)^2 - (\text{inner} \ \ r)^2\right) \, d\theta$$

$$= \boxed{\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin\theta)^2 - (1)^2 \; d\theta}$$

OR using symmetry

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\theta)^2 - (1)^2 \; d\theta\right)$$

(c) The **area** that lies inside both of the curves $r = 1+\sin\theta$ and inside the polar curve $r = 1-\sin\theta$.



These two polar curves intersect when $\theta = 0$ and $\theta = \pi$.

Using symmetry, we have

Area $= A = 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta\right)$

$$= \boxed{4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\sin\theta)^2 \ d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1-\sin\theta)^2 \ d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{2\left(\frac{1}{2}\int_0^{\pi} (1-\sin\theta)^2 \ d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (1+\sin\theta)^2 \ d\theta\right)}$$

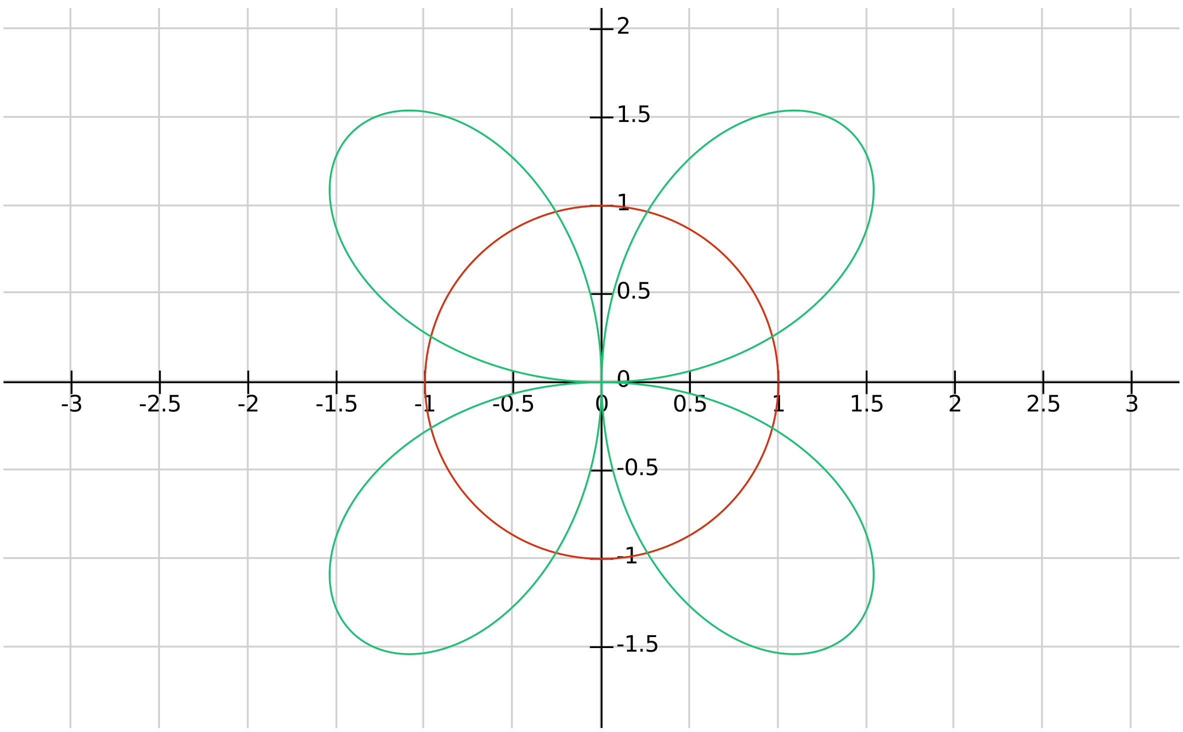
OR you could use symmetry again

$$A = \boxed{4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (1+\sin\theta)^2 \ d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{2\left(\frac{1}{2}\int_{\pi}^{2\pi} (1+\sin\theta)^2 \ d\theta\right)}$$

(d) The **area** bounded outside the polar curve $r = 1$ and inside the polar curve $r = 2\sin(2\theta)$.



These two polar curves intersect when $2\sin(2\theta) = 1 \Rightarrow \sin(2\theta) = \frac{1}{2} \Rightarrow \theta = \pm\dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$.

Using symmetry

$$\text{Area} = A = 4\left(\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left((\text{outer} \;\; r)^2 - (\text{inner} \;\; r)^2\right) \; d\theta\right)$$

$$= \boxed{4\left(\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta)^2 - (1)^2 \ d\theta\right)}$$

OR using more symmetry

$$A = 8\left(\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (2\sin(2\theta)^2 - (1)^2 \ d\theta\right)$$