Math 121 Final Exam May 13, 2015

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(1-x)+x}{\cosh(4x)-\arctan(3x)-e^{-3x}}$ (b) $\displaystyle\lim_{x\to\infty} \left(e^{\frac{1}{x^3}} - \frac{5}{x^3}\right)^{x^3}$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{x^4+3x^3+6x^2+6x+5}{x^3+x^2+2x+2}\,dx = \int \frac{x^4+3x^3+6x^2+6x+5}{(x+1)(x^2+2)}\,dx$ (b) $\displaystyle\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16-x^2}}\,dx$

(c) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left[1+\sin^2 x\right]^{\frac{7}{2}}}\,dx$

3. [25 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_6^{\infty} \frac{1}{x^2-10x+28}\,dx$ (b) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}\cdot\arcsin x}\,dx$

(c) $\displaystyle\int_1^{\infty} \frac{1}{x^2+5x+6}\,dx$ (d) $\displaystyle\int_0^{1} x\ln x\,dx$

4. [15 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,7^{n+1}}{3^{3n-1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2^{n+1}\,(\ln 5)^n}{n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n-1}}{3\,(2n)!}$

(d) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$ (e) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(n^3+7)}{n^7+3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\arctan(7n)}{n^7+7}$ (c) $\displaystyle\sum_{n=1}^{\infty} n\cdot\arcsin\left(\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,e^{3n}\,(2n)!}{n^n\,4^{2n}\,(n!)^2}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,n}{n^2+4}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (\ln n)\ (4x-1)^n}{n^2 \cdot 5^n}$. Analyze carefully and with full justification.

7. [8 Points]

(a) Write the MacLaurin Series for $f(x) = x^4 \arctan(2x)$. State the Radius of Convergence for this series.

(b) Use this series to determine the **seventh**, **eighth** and **ninth** derivatives of $f(x) = x^4 \arctan(2x)$ evaluated at $x = 0$. Do **Not** Simplify your answers here in part (b).

8. [12 Points] Please analyze with detail and justify carefully. Simplify your answers.

(a) Estimate $e^{-\frac{1}{3}}$ with error less than $\frac{1}{100}$. Justify in words that your error is indeed less than $\frac{1}{100}$.

(b) Estimate $\arctan\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$. Justify in words that your error is indeed less than $\frac{1}{100}$.

(c) Estimate $\cos(1)$ with error less than $\frac{1}{10}$. Justify in words that your error is indeed less than $\frac{1}{10}$.

9. [15 Points]

(a) Consider the region bounded by $y = e^x - 1,\ y = 3,\ x = 0$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = -1}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(b) Consider the region bounded by $y = \arcsin x,\ y = 1$, and $x = 0$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = 5}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(c) Consider the region bounded by $y = \arctan x,\ y = 4,\ x = 0$ and $x = 1$. Rotate the region about the $\boxed{y\text{-axis}}$. **COMPUTE** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [15 Points]

(a) Consider the Parametric Curve represented by $x = (\arctan t) - t$ and $y = 2\sinh^{-1} t$.
COMPUTE the **arclength** of this parametric curve for $0 \le t \le \sqrt{3}$.

Recall $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$

(b) Consider a *different* Parametric Curve represented by $x = t + \frac{1}{t}$ and $y = \ln\left(t^2\right)$. **COMPUTE** the **surface area** obtained by rotating this curve about the $\boxed{y\text{-axis}}$, for $1 \le t \le 2$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 1 + \sin\theta}$ and inside the polar curve $\boldsymbol{r = 3\sin\theta}$. **Sketch** the Polar curves **and** shade the bounded area.