Math 121 Final Exam December 20, 2015

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle \lim_{x\to\ln 3} \frac{3-e^x}{e^{-2x}-\frac{1}{9}}$ (b) $\displaystyle \lim_{x\to 0} \frac{\ln(1-x)+\arctan x}{xe^x-\sinh x}$ (c) $\displaystyle \lim_{x\to\infty} \left(1-\arcsin\left(\frac{6}{x}\right)\right)^x$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle \int \frac{x^5}{\sqrt{4-x^2}}\,dx$ (using a trigonometric substitution) (b) $\displaystyle \int_1^3 \frac{1}{\sqrt{x}\,(x+3)}\,dx$

(c) $\displaystyle \int_e^{e^{\sqrt{5}}} \frac{1}{x(4+(\ln x)^2)^{\frac{3}{2}}}\,dx$ (d) $\displaystyle \int x\arcsin x\,dx$

3. [24 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle \int_1^2 \frac{4}{x^2-8x+12}\,dx$ (b) $\displaystyle \int_{-\infty}^{\infty} \frac{1}{x^2-8x+19}\,dx$

(c) $\displaystyle \int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\ \ln x\,dx$

4. [18 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^n\ 4^{2n+1}}{3^{3n-1}}$ (b) $\displaystyle \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ 2^{n+1}\ (\ln 6)^n}{n!}$ (c) $\displaystyle \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{4n}\ (2n)!}$

(d) $\displaystyle -\frac{1}{5}+\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}+\frac{1}{4\cdot 5^4}-\dots$ (e) $\displaystyle 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots$ (f) $\displaystyle \sum_{n=0}^{\infty} \frac{1}{e^n}$ (g) $\displaystyle \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(36)^n\ (2n+1)!}$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^n\ (n^4+7)}{n^7+4}$ (b) $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^n \arctan(7n)}{e^n+7}$ (c) $\displaystyle \sum_{n=1}^{\infty} n\cdot\arctan\left(\frac{1}{n}\right)$

(d) $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^n\ \sqrt{n}}{n+3}$ (e) $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ e^{3n}\ (3n)!}{n^n\ 4^{2n}\ (n!)^2}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ (\ln n) \ (4x-1)^n}{n^2 \cdot 5^n}$. Analyze carefully and with full justification.

7. [8 Points]

(a) Write the MacLaurin Series for the hyperbolic cosine $\boldsymbol{f(x) = \cosh x}$.

(b) Write the MacLaurin Series for $\boldsymbol{f(x) = \cosh(2x^3)}$.

(c) Use this series to determine the **twelfth**, and **thirteenth**, derivatives of $\boldsymbol{f(x) = \cosh(2x^3)}$ evaluated at $\boldsymbol{x = 0}$. That is, compute $\boldsymbol{f^{(12)}(0)}$ and $\boldsymbol{f^{13}(0)}$. Do **not** simplify your answers here.

8. [12 Points] Please analyze with detail and justify carefully. Simplify your answers.

(a) Use the MacLaurin series representation for $\boldsymbol{f(x) = x \sin\left(x^2\right)}$ to **Estimate** $\displaystyle\boldsymbol{\int_0^1 x \sin\left(x^2\right) \, dx}$ with error less than $\boldsymbol{\frac{1}{100}}$. Justify in words that your error is less than $\boldsymbol{\frac{1}{100}}$.

(b) Estimate $\boldsymbol{\cos\left(\frac{1}{2}\right)}$ with error less than $\boldsymbol{\frac{1}{100}}$. Justify in words that your error is indeed less than $\boldsymbol{\frac{1}{100}}$.

9. [10 Points] Consider the region bounded by $\boldsymbol{y = \cos x}$, $\boldsymbol{y = x + 1}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = \frac{\pi}{2}}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = 3}}$. **COMPUTE** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [18 Points]

(a) Consider the Parametric Curve represented by $\boldsymbol{x = t + \frac{1}{1+t}}$ and $\boldsymbol{y = 2\ln(1+t)}$.

COMPUTE the **arclength** of this parametric curve for $\boldsymbol{0 \leq t \leq 4}$.

(b) Consider a *different* Parametric Curve represented by $\boldsymbol{x = t - e^{2t}}$ and $\boldsymbol{y = 1 - \sqrt{8}\, e^t}$. **COMPUTE** the **surface area** obtained by rotating this curve about the $\boxed{\boldsymbol{y}\text{-axis}}$, for $\boldsymbol{0 \leq t \leq 3}$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 1 + \sin\theta}$ and inside the polar curve $\boldsymbol{r = 3\sin\theta}$. **Sketch** the Polar curves **and** shade the bounded area.