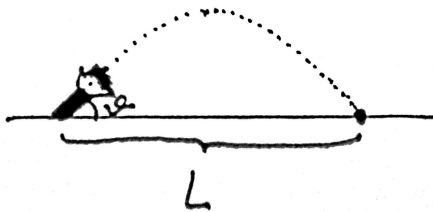


Optimization examples

November 13, 2017

Problem: (adapted from an example in class)

If a cannonball is launched from the ground at an initial speed of v_0 meters per second, at an angle θ with the ground, the total distance it travels before hitting the ground is given by the following formula (you are not responsible to know this formula, but it is a good exercise to derive it if you know a little physics). Here g is the constant 9.8 meters per second per second (m/s^2), which measures the acceleration due to gravity on the surface of the earth.



$$L = \frac{2v_0^2}{g} \sin \theta \cos \theta$$

What launch angle θ causes the cannon ball to fly the furthest possible distance? Does it matter what the value of g or v_0 is (for example, is the best launch angle the same on the moon)?

Solution:

Regard v_0, g as constants, and think of the total distance as a function $L(\theta)$ of θ . Then θ must be chosen in the interval $[0, \pi/2]$ (the two endpoints being shooting the cannon straight forward or shooting it straight up into the air). So let's maximize this function on the interval $[0, \pi/2]$. Begin by finding $L'(\theta)$.

$$\begin{aligned} L(\theta) &= \frac{2v_0^2}{g} \sin \theta \cos \theta \\ L'(\theta) &= \frac{2v_0^2}{g} [(\sin \theta)' \cos \theta + \sin \theta (\cos \theta)'] \text{ (Product rule)} \\ &= \frac{2v_0^2}{g} [\cos \theta \cos \theta - \sin \theta \sin \theta] \text{ (Derivatives of sine and cosine)} \\ &= \frac{2v_0^2}{g} (\cos^2 \theta - \sin^2 \theta) \end{aligned}$$

Now solve $L'(\theta) = 0$ to find the critical numbers.

$$\begin{aligned}
0 &= \frac{2v_0^2}{g} (\cos^2 \theta - \sin^2 \theta) \\
\Leftrightarrow 0 &= \cos^2 \theta - \sin^2 \theta \\
\Leftrightarrow 0 &= (\cos \theta + \sin \theta)(\cos \theta - \sin \theta) \\
\Leftrightarrow \cos \theta &= \pm \sin \theta
\end{aligned}$$

Now, on the interval $[0, \frac{\pi}{2}]$, both $\sin \theta$ and $\cos \theta$ are positive, so $\cos \theta = \pm \sin \theta$ means that $\cos \theta = \sin \theta$, which occurs only at the angle $\theta = \frac{\pi}{4}$ (45 degrees). So this is the only critical number.

Evaluating the function at the critical number and endpoints gives:

$$\begin{aligned}
L(0) &= 0 \\
L(\pi/4) &= \frac{2v_0^2}{g} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = v_0^2/g \\
L(\pi/2) &= 0
\end{aligned}$$

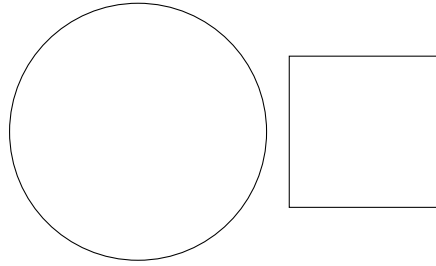
So the absolute maximum occurs at the critical number $\boxed{\theta = \pi/4}$. So the optimal launch angle is 45 degrees.

Notice that the optimal launch angle θ **does not depend on the initial velocity, or the strength of gravity**. The furthest possible distance does depend on these, but the launch angle does not. As you would expect, weaker gravity makes the cannonball fly further, as does a faster muzzle velocity.

Note. If you happen to know the double-angle formula for $\sin(2\theta)$, this problem can be solved somewhat more easily. We are not assuming knowledge of this formula however.

Problem:

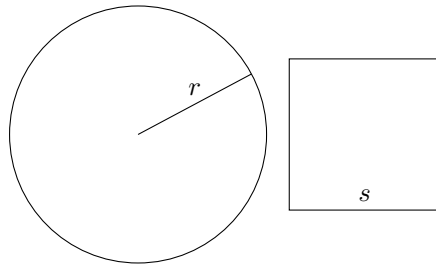
Suppose that you clip a single piece of wire, 10cm long, into two pieces. One piece you bend into a circle, and the other you bend into a square.



Consider the total area enclosed by the circle and the square together. What is the **maximum** total area that can be enclosed in this way?

Solution:

First introduce two new variables r and s to denote the radius of the circle and the side length of the square, in centimeters.



Then the total area is given by $\pi r^2 + s^2$, and the constraint that the original wire was 10 centimeters long can be expressed by $2\pi r + 4s = 10$ (this just says that the perimeters of the two figures add up to 10). Solve for one variable to obtain $s = \frac{5 - \pi r}{2}$. Now substitute this back to obtain the area as a function of the other variable, r .

$$\begin{aligned} A(r) &= \pi r^2 + \left(\frac{5 - \pi r}{2}\right)^2 \\ &= \left(\pi + \frac{1}{4}\pi^2\right)r^2 - \frac{5\pi}{2}r + \frac{25}{4} \end{aligned}$$

The interval of possible values of r begins at 0 (this is the case where you don't clip the wire, and only make a square) and ends at $\frac{5}{\pi}$ (this is the case where you only make a circle of circumference 10). So the relevant interval is $[0, \frac{5}{\pi}]$.

In this case, we could take the derivative and find the critical numbers as usual. But in this particular case, we can actually skip the algebra. The reason is that the second derivative of this function is $A''(r) = 2\left(\pi + \frac{1}{4}\pi^2\right)$, which is positive. So this function is concave up everywhere. In particular, any critical numbers are local *minima*, not local maxima. So *the absolute maximum must occur at an endpoint*. To see which endpoint, just check the value of the function at the endpoints.

$$\begin{aligned} A(0) &= \frac{25}{4} \\ A\left(\frac{5}{\pi}\right) &= \frac{25}{\pi} \end{aligned}$$

The larger of the two of these is $\boxed{25/\pi \approx 7.96}$. So this is maximum possible area: it is achieved when you turn the entire wire into a circle, and none of it into a square.

If you want to check your work: the critical number here is in fact $\frac{5}{4+\pi} \approx 0.61$, and it gives the absolute minimum, which is approximately 3.5. That is the *minimum* possible total area of the square and rectangle.

Problem:

Find all extrema of the function $f(x) = \sin x + \sin^2 x$ on the interval $[0, 2\pi]$. Use these, along with any x -intercepts, to make a rough sketch of the graph.

Solution:

First, compute the derivative of this function.

$$\begin{aligned} f(x) &= \sin x + \sin x \cdot \sin x \\ f'(x) &= (\sin x)' + 2 \sin x (\sin x)' \\ &= \cos x + 2 \sin x \cos x \\ &= \cos x(1 + 2 \sin x) \end{aligned}$$

Comment on strategy: I have factored this expression in the last line. This is often prudent in optimization, since I will be looking for places where it is equal to 0.

To find the critical points, solve $0 = \cos x(1 + 2 \sin x)$. This equation holds precisely when either $\cos x = 0$ or $1 + 2 \sin x = 0$. That is, either $\cos x = 0$ or $\sin x = -\frac{1}{2}$. Solving these two equations separately gives the critical numbers $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$. Arranged in increasing order, these are $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

To determine which of these are which type of extremum, I will use the *second derivative test*. (Note that it would also be possible to use the first derivative test, or a couple other methods).

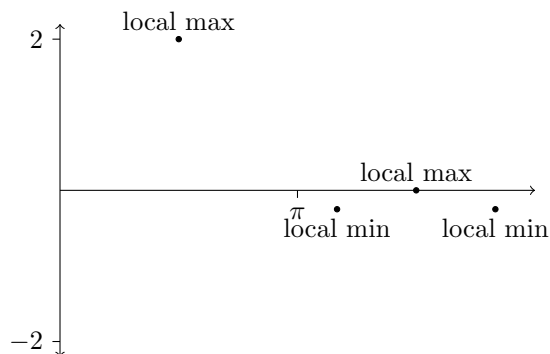
$$\begin{aligned} f''(x) &= -\sin x + 2 \cos x \cos x - 2 \sin x \sin x \\ &= -\sin x + 2 \cos^2 x - 2 \sin^2 x \end{aligned}$$

This will be easier to study if it is expressed using only sines, so substitute $\cos^2 x = 1 - \sin^2 x$ to obtain the following expression.

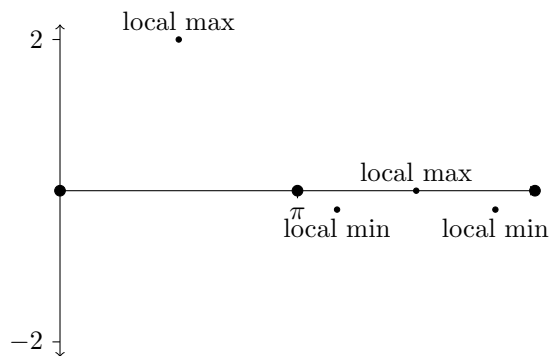
$$\begin{aligned} f''(x) &= -\sin x + 2 - 2 \sin^2 x - 2 \sin^2 x \\ &= -4 \sin^2 x - \sin x + 2 \end{aligned}$$

Evaluating this at the four critical numbers gives $f''(\frac{\pi}{2}) = 2, f''(\frac{7\pi}{6}) = \frac{3}{2}, f''(\frac{3\pi}{2}) = -1, f''(\frac{11\pi}{6}) = 2$. Thus these are, respectively: a local max, local min, local max, local min.

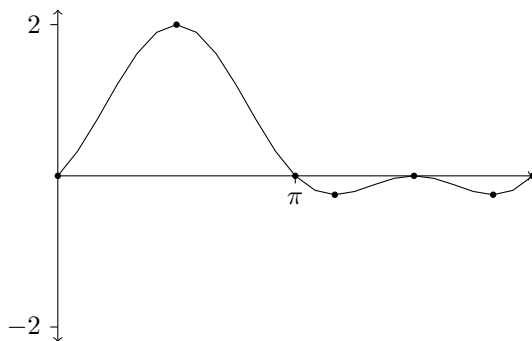
Computing the value of the function at these critical points gives $f(\frac{\pi}{2}) = 2, f(\frac{7\pi}{6}) = -\frac{1}{4}, f(\frac{3\pi}{2}) = 0, f(\frac{11\pi}{6}) = -\frac{1}{4}$. So begin by plotting these four extrema.



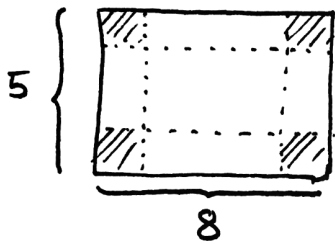
Next, let's find the x -intercepts. Notice that $f(x) = 0$ if and only if $\sin x + \sin^2 x = 0$, i.e. $\sin x(1 + \sin x) = 0$, i.e. $\sin x = 0$ or $\sin x = -1$. This gives x values $0, \pi, \frac{3\pi}{2}$, and 2π . So add these points to the plot also.



These are enough points to get a pretty good idea of the curve's shape. It is plotted below.

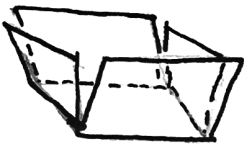


Problem

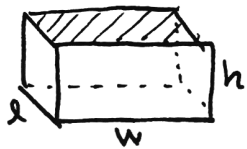


A box with no top is to be constructed by cutting four square corners from an 8x5 piece of cardboard & folding up the flaps.

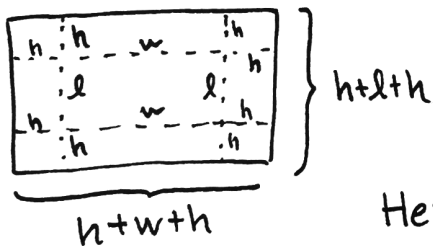
What is the maximum volume of such a box?



Solution



Suppose the eventual dimensions of the box are h, w, l as shown. Labeling these on the unfolded box gives the following constraints:



$$l+2h = 5$$

$$w+2h = 8$$

Hence we can solve for l, w & write the volume as a function of h alone:

$$V(h) = h \cdot (5-2h)(8-2h).$$

We want the maximum for h chosen from $[0, 5/2]$, since this ensures h, w, l are all ≥ 0 .

Now,

$$\begin{aligned}V(h) &= h \cdot (4h^2 - 26h + 40) \\ &= 4h^3 - 26h^2 + 40h\end{aligned}$$

$$\begin{aligned}\Rightarrow V'(h) &= 12h^2 - 52h + 40 \\ &= 4 \cdot (3h^2 - 13h + 10) \\ &= 4 \cdot (3h - 10)(h - 1)\end{aligned}$$

so the possible critical numbers are $h=1$ & $h=10/3$. But $10/3$ is out of range ($10/3 > 5/2$), so only $h=1$ is relevant.

There are now several ways to check whether $h=1$ gives a maximum. Here are a few:

1) closed interval method.

$$\left. \begin{aligned}V(0) &= 0 \cdot (5 - 2 \cdot 0) \cdot (8 - 2 \cdot 0) = 0 \\ V(1) &= 1 \cdot 3 \cdot 6 = 18 \\ V(5/2) &= 5/2 \cdot 0 \cdot 3 = 0\end{aligned} \right\} \begin{array}{l} \text{of these candidates, } 18 \\ \text{is largest} \\ \Rightarrow \text{maximum is } 18, \\ \text{at } \underline{h=1}. \end{array}$$

2) sign chart

	4	$3h-10$	$h-1$	$V'(h)$
$0 < h < 1$	+	-	-	+
$1 < h < 5/2$	+	-	+	-

} hence $h=1$ gives a local max.
It's the only one, hence a global max on $[0, 5/2]$

3) second deriv. test.

$$\begin{aligned}V''(h) &= 24h - 52 \\ V''(1) &= -28 < 0 \quad (\text{conc. down}) \quad \curvearrowright\end{aligned}$$

so $h=1$ gives a local max. (func. changes from inc. to dec. there)

so $h=1$ gives the maximum volume.

4) Test values. $V'(\frac{1}{2}) = 4 \cdot (\frac{3}{2} - 10) \cdot (-\frac{1}{2}) = 4 \cdot (-\frac{17}{2}) \cdot (-\frac{1}{2}) = \frac{4 \cdot 17}{4} > 0$

and $v'(2) = 4 \cdot (6-10) \cdot (2-1) = 4(-4) \cdot 1 = -16 < 0$.

So $v(h)$ is inc. on $(0,1)$ & dec on $(1, \infty)$.

$\Rightarrow h=1$ gives the max.