Read This First!

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted. Cell phones are to be out of sight.

- Please read each question carefully. Show all work clearly in the space provided.

- If you need addition space to do a problem, please use the back of the previous page.

- In order to receive full credit on a problem, solution methods must be complete, logical and understandable

- Answers must be clearly labeled.

- The exam consists of Questions 1–9, which total to 100 points.

Grading - For Instructor Use Only

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1. [12 points] Let \( f(x) = 3x^5 - 20x^3 \).
   
   (a) Find the critical numbers of \( f(x) \).
   
   \[
   f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x-2)(x+2)
   \]
   
   \[= 0 \text{ when } x = 0 \text{ or } x^2 = 4, \text{ & never undefined} \]
   
   \[\text{i.e. crit. numbers are } \{0, -2, 2\}. \]
   
   (b) Test whether they are local maximums, local minimums, or neither.
   
   \[
   \begin{array}{cccc}
   -2 & 0 & 2 & \\
   15x^2 & + & + & + \\
   x-2 & - & - & + \\
   x+2 & - & + & + \\
   \hline
   f' & + & - & - & + \\
   f & & & & \\
   \end{array}
   \]
   
   by first deriv. test,
   
   There's a \boxed{\text{local max at } x = -2} \& \boxed{\text{local min at } x = 2}.
   
   (no max/min at \( x = 0 \))

2. [10 points] Consider a function \( g(x) \) with the property that \( g(2) = 3, g'(2) = 0, \) and \( g''(2) \) is some nonzero number. We are also given that \( g(x) \) has a local maximum when \( x = 2 \). Determine whether \( g''(2) \) positive is negative. Your solution should include an explanation (in words!) and a picture. Be sure to indicate how the words relate to the picture.

   Because there's a local max \( @ \ x = 2, \) the second deriv. test says that \( f''(2) < 0 \) (since it's nonzero).

   In other words, the graph must be concave down \( @ \ x = 2 \) rather than concave up at a local maximum.
3. [16 points] The radius and height of a cylinder are changing with respect to time. The radius is increasing at a rate of 2 cm/sec, while the height is decreasing at a rate of 1 cm/sec. How fast is the volume of the cylinder changing at the instant of time when the radius is 10 cm and the height is 5 cm? (You may assume that the volume of a cylinder of radius \( r \) and height \( h \) is \( V = \pi r^2 h \)).

\[
V = \pi r^2 h.
\]

\[
\Rightarrow V' = \pi \cdot 2r \cdot r' h + \pi r^2 h'.
\]

Substituting values at key moment:

\[
V' = \pi \cdot 2 \cdot 10 \cdot 2 \cdot 5 + \pi \cdot 10^2 \cdot (-1)
\]

\[
= 200\pi - 100\pi
\]

\[
= [100\pi] \text{ (cm/sec)}
\]
4. [14 points] Find the absolute maximum and minimum values of the function

\[ f(x) = x(x - 4)^3 \]

on the interval [0, 3].

\[ f'(x) = 1 \cdot (x-4)^3 + x \cdot 3(x-4)^2 \]
\[ = (x-4)^2 [(x-4) + 3x] \]
\[ = (x-4)^2 [4x-4] \]
\[ = 4(x-4)^2(x-1) \]

This is never undefined.
It is 0 @ x=1 & x=4; only x=4 is in [0,3].

Candidate: \(x=0, x=1, x=3\).

\[ f(0) = 0 \cdot (-4)^3 = 0 \leq \text{max} \]
\[ f(1) = 1 \cdot (-3)^3 = -27 \leq \text{min} \]
\[ f(3) = 3 \cdot (-1)^3 = -3 \]

max value is 0 @ x=0
min value is -27 @ x=1
5. [16 points] Find where $g(x) = \frac{x}{(x+3)^2}$ is increasing and decreasing.

$$g'(x) = \frac{1 \cdot (x+3)^2 - x \cdot 2(x+3)}{[(x+3)^2]^2}$$

After simplification:

$$= \frac{(x+3)^4 - 2x(x+3)}{(x+3)^6} = \frac{3-x}{(x+3)^3}$$

The sign could change at $x = 3$ on $x = -3$ (num. 0 or denom. 0).

Decreasing on $(-\infty, -3) \cup (3, \infty)$
Increasing on $(-3, 3)$
6. [20 points] The function \( f(x) = \frac{1 + x^2}{x^2 - 4} \) has first and second derivatives given by:

\[
f'(x) = \frac{-10x}{(x^2 - 4)^2}, \quad f''(x) = \frac{10(3x^2 + 4)}{(x^2 - 4)^3}.
\]

(a) Use this information to determine where \( y = f(x) \) is increasing or decreasing, and find any local max(s) or local min(s).

Sign of \( f' \) could change at \( x = \pm 2 \) on \( 0 \) (num. or denom 0)

Note: \( x = \pm 2 \) are discontinuities of \( f(x) \), so they cannot be local max(s) or min.

Increasing: \((-\infty, -2) \cup (-2, 0)\)
Decreasing: \((0, 2) \cup (2, \infty)\)
Local max \( @ x = 0 \)

(b) Use this information to determine where \( y = f(x) \) is concave up or concave down.

Sign of \( f''(x) \) could change only at \( x = \pm 2 \)
(when denom \( \neq 0 \), since \( 3x^2 + 4 > 0 \) for all \( x \).

\[
can \text{ factor:} \quad \frac{10(3x^2 + 4)}{(x^2 - 4)^3} = \frac{10(3x^2 + 4)}{(x+2)(x-2)^2} = \frac{10(3x^2 + 4)}{(x+2)(x-2)}
\]

Conc. up on \((-\infty, -2) \cup (2, \infty)\)
Conc. down on \((-2, 2)\)
(c) Compute \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). Use these values to identify any horizontal asymptotes of the graph \( y = f(x) \).

\[
\lim_{x \to \infty} \frac{1+x^2}{x^2-4} \cdot \frac{\sqrt{x^2}}{\sqrt{x^2}} = \lim_{x \to \infty} \frac{\sqrt{x^2} + 1}{1 - 4/x^2} = \frac{\infty + 1}{1 - 4/\infty^2} = \frac{\infty + 1}{1 - 0} = \infty
\]

\( \text{Hor. asymptote at } y = 1 \)

\[
\lim_{x \to -\infty} \frac{1+x^2}{x^2-4} = \lim_{x \to -\infty} \frac{\sqrt{x^2} + 1}{1 - 4/x^2} = \frac{\sqrt{(-\infty)^2} + 1}{1 - 4/\infty^2} = \frac{-\infty + 1}{1 - 0} = \frac{-\infty + 1}{1 - 0}
\]
7. [12 points] The previous problem stated that $f(x) = \frac{1 + x^2}{x^2 - 4}$ has derivatives

$$f'(x) = \frac{-10x}{(x^2 - 4)^2}, \quad f''(x) = \frac{10(3x^2 + 4)}{(x^2 - 4)^3}.$$ 

Verify that these formulas are correct by computing $f'(x)$ and $f''(x)$ using the usual rules of differentiation.

$$f'(x) = \frac{2x(x^2-4) - (1+x^2)2x}{(x^2-4)^2}$$
$$= \frac{2x^2 - 8x - 2x^3 - 2x^3}{(x^2-4)^2}$$
$$= \frac{-10x}{(x^2-4)^2} \checkmark$$

$$f''(x) = \frac{-10 \cdot (x^2-4)^2 - (-10x) \cdot 2(x^2-4)2x}{[(x^2-4)^2]^2}$$
$$= \frac{(x^2-4) \cdot [-10(x^2-4) + 40x^2]}{(x^2-4)^4}$$
$$= \frac{10 \cdot [-x^2 + 4 + 4x^2]}{(x^2-4)^3}$$
$$= \frac{10 \cdot (3x^2+4)}{(x^2-4)^3} \checkmark$$