Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show ALL work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

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1. [21 points] Evaluate each limit.

(a) \[ \lim_{x \to 7} \frac{2}{6x - 7} = \frac{2}{7} \cdot \frac{7-1}{7+1} = \frac{6}{8} = \frac{3}{4} \]

(b) \[ \lim_{x \to 1} \frac{\sqrt{4 - 3x} - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(\sqrt{4 - 3x} - 1) \cdot (\sqrt{4 - 3x} + 1)}{(x+1)(x-1)(\sqrt{4 - 3x} + 1)} = \frac{-3}{2 \cdot (\sqrt{2} + 1)} = -\frac{3}{4} \]

(c) \[ \lim_{x \to 4^-} \frac{|x - 4|}{x^2 - 3x - 4} \]

\[ |x-4| = -(x-4) \quad \text{for} \quad x \leq 4, \quad \text{so} \] \[ \lim_{x \to 4^-} \frac{-(x-4)}{(x+1)(x-4)} = \frac{-1}{5} = -\frac{1}{5} \]
(d) \[ \lim_{x \to -\infty} \frac{3 - 4x - 4x^2}{9 + 2x^2} \]

\[ = \lim_{x \to -\infty} \frac{\frac{3}{x^2} - \frac{4}{x} - 4}{\frac{9}{x^2} + 2} \cdot \frac{1}{x^2} \]

\[ = \lim_{x \to -\infty} \frac{3}{9} + \frac{4}{x} - 4 = \frac{3}{\infty} - \frac{4}{(-\infty)} - 4 \]

\[ = -\frac{4}{2} = -2 \]

(e) \[ \lim_{x \to \infty} \frac{x^3 - 10x^2}{5x^2 + 7} \]

\[ = \lim_{x \to \infty} \frac{x^2 - 10x^2}{5x^2 + 7} \cdot \frac{1}{x^2} = \lim_{x \to \infty} \frac{x - 10}{5 + \frac{7}{x^2}} \]

\[ = \frac{\infty - 10}{5 + \frac{7}{\infty}} = \frac{\infty - 10}{\infty} = +\infty \]
(f) \[ \lim_{{x \to 8}} \frac{{x + 4 \cdot x - 2 - (x - 4) \cdot x + 4}}{x^2 - 10x + 16} \]
\[= \lim_{{x \to 8}} \frac{x^2 - 2x - x^2 + 16}{(x + 4)(x - 2)} \]
\[= \lim_{{x \to 8}} \frac{-2(x - 8)}{(x + 4)(x - 2)(x - 8)} \]
\[= \lim_{{x \to 8}} \frac{-2}{(x + 4)(x - 2)} \]
\[= \frac{-2}{6.6.6} = \frac{-1}{216} \]

(g) \[\lim_{{x \to 2^+}} \frac{{x^2 - x - 6}}{x^2 + x - 6} \]
\[= \lim_{{x \to 2^+}} \frac{(x+2)(x-3)}{(x+2)(x+3)} \]
\[= \frac{4 \cdot (-1)}{0^+ \cdot 5} = \frac{-4}{0^+} = \infty \]
2. [15 points] Evaluate the derivative of each function. You do not need to simplify your answer.

(a) \( f(x) = (3x - 7) \left( x^{1/3} + \frac{1}{x^4} \right) \)

\[
\begin{align*}
\frac{d}{dx} f(x) &= 3 \cdot \left( x^{1/3} + \frac{1}{x^4} \right) + (3x - 7) \cdot \left( \frac{1}{3} x^{-2/3} - 4 \cdot \frac{1}{x^5} \right) \\
&= \frac{d}{dx} \left( x^{1/3} + \frac{1}{x^4} \right) + (3x - 7) \cdot \left( \frac{1}{3} x^{-2/3} - 4 \cdot \frac{1}{x^5} \right)
\end{align*}
\]

(b) \( g(x) = (2x^3 + 5x^4)^{1/3} \)

\[
\begin{align*}
\frac{d}{dx} g(x) &= \frac{1}{3} \cdot (2x^3 + 5x^4)^{-2/3} \cdot (6x^2 + 20x^3) \\
&= \frac{1}{3} \cdot (2x^3 + 5x^4)^{-2/3} \cdot (6x^2 + 20x^3)
\end{align*}
\]

(c) \( h(x) = (x^2 + 7)^{\sqrt{5x+3}} \)

\[
\begin{align*}
\frac{d}{dx} h(x) &= 2x \cdot \sqrt{5x+3} + (x^2 + 7) \cdot \frac{1}{2\sqrt{5x+3}} \cdot 5 \\
&= 2x \cdot \sqrt{5x+3} + (x^2 + 7) \cdot \frac{1}{2\sqrt{5x+3}} \cdot 5
\end{align*}
\]
(d) \( f(x) = \frac{\sqrt{2x+3}}{x^2 + 1} \)

\[
    f'(x) = \frac{\frac{1}{2\sqrt{2x+3}} \cdot (2x^2 + 1) - \sqrt{2x+3} \cdot 2x}{(x^2 + 1)^2}
\]

(e) \( g(x) = (2x - 1)^3(5x + 3)^5 \)

\[
    g'(x) = 3(2x-1)^2 \cdot 2 \cdot (5x+3)^5 + (2x-1)^3 \cdot 5(5x+3)^4 \cdot 5
\]
3. [9 points] Let \( f(x) = \frac{2x}{3x + 1} \). Compute \( f'(x) \) using the limit definition of the derivative. You may use the quotient rule to check your answer, but for full points all steps of the limit calculation must be shown.

\[
\begin{align*}
\frac{f'(x)}{h} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{2(x+h)}{3(x+h)+1} - \frac{2x}{3x+1}}{h} \\
&= \lim_{h \to 0} \frac{\frac{2(x+h)}{3(x+h)+1} \cdot \frac{3x+1}{3x+1} - \frac{2x}{3x+1} \cdot \frac{3(x+h)+1}{3(x+h)+1}}{h} \\
&= \lim_{h \to 0} \frac{\frac{6x^2 + 2x + 6xh + 2h - 6x^2 - 2xh - 2x}{h(3(x+h)+1)(3x+1)}}{h} \\
&= \lim_{h \to 0} \frac{2h}{h(3(x+h)+1)(3x+1)} \\
&= \frac{2}{(3(x+h)+1)(3x+1)} \\
&= \frac{2}{(3x+1)^2}
\end{align*}
\]
4. Consider the curve defined by the equation

\[ y^2 = x^3 - x + 1. \]

(a) [4 points] Determine \( \frac{dy}{dx} \) using implicit differentiation. Your answer will be in terms of both \( x \) and \( y \).

\[
\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - x + 1)
\]

\[
2y \cdot \frac{dy}{dx} = 3x^2 - 1
\]

\[
\frac{dy}{dx} = \frac{3x^2 - 1}{2y}
\]

(b) [4 points] Find the equation of the tangent line at the point \((3, 5)\).

\[ @ (3, 5), \quad \frac{dy}{dx} = \frac{3 \cdot 3^2 - 1}{2 \cdot 5} = \frac{26}{10} = \frac{13}{5} \]

So the tangent line is

\[ y - 5 = \frac{13}{5} (x - 3) \]

ie.

\[ y = \frac{13}{5} x - \frac{39}{5} + 5 \]

ie.

\[ y = \frac{13}{5} x - \frac{14}{5} \]
5. [9 points] Find the absolute maximum and absolute minimum values of \( f(x) = x^2(x - 5)^3 \) on the interval \([0, 6]\).

\[
\begin{align*}
\frac{d^2}{dx} f(x) &= 2x(x-5)^3 + x^2 \cdot 3(x-5)^2 \\
&= x(x-5)^2 \cdot [2(x-5) + 3x] \\
&= x(x-5)^2(3x + 5 - 10) \\
&= 5x(x-5)^2(x-2) \\
\Rightarrow \text{crit. numbers} &\quad 0, 2, 5.
\end{align*}
\]

**Closed interval method:**

\[
\begin{align*}
f(0) &= 0^2(-5)^3 = 0 \\
f(2) &= 2^2(-3)^3 = 4 \cdot (-27) = -108 \quad \leftarrow \text{min} \\
f(5) &= 5^2 \cdot 0^3 = 0 \\
f(6) &= 6^2 \cdot 1^3 = 36 \quad \leftarrow \text{max}
\end{align*}
\]

**abs. max. is 36 @ x = 6**

**abs. min. is -108 @ x = 2.**
6. [12 points] Consider the following function.

\[
f(x) = \frac{2 - x}{x^2}
\]

Sketch the graph \( y = f(x) \). Clearly label the following features on your graph: asymptotes (horizontal or vertical), intervals where it is increasing/decreasing, intervals where it is concave up/down, local max(s) and min(s), and inflection point(s).

Division by 0 occurs @ \( x = 0 \), & numerator a \( z-0 \neq 0 \) then

\[
\lim_{x \to \infty} \frac{2-x}{x^2} = \lim_{x \to 0} \left( \frac{\frac{2}{x} - \frac{1}{x^2}}{1} \right) = \frac{2}{\infty} - \frac{1}{\infty} = 0 \}
\]

so \( H.A. \) @ \( y = 0 \)

\[
\mathcal{f}'(x) = \left( \frac{2}{x^3} - \frac{1}{x^2} \right) = - \frac{4}{x^3} - (-1) \frac{1}{x^2} = \frac{-4}{x^3} + \frac{1}{x^2} = \frac{x-4}{x^3} \]

\[
\text{num } 0 \cap x = 4 \]

\[
\text{denom } 0 \cap x = 0
\]

\[
\mathcal{f}''(x) = \left( \frac{-4}{x^3} + \frac{1}{x^2} \right) = \frac{12}{x^4} - \frac{2}{x^3} = \frac{12-2x}{x^4} = -2 \cdot \frac{x-6}{x^4} \]

\[
\text{num } 0 \cap x = 6
\]

\[
\text{denom } 0 \cap x = 0
\]

\[
\text{inc on } (-\infty, 0) \& (4, \infty)
\]

\[
\text{dec on } (0, 4)
\]

Local min @ \( x = 4 \), \( y = f(4) = -1/8 \)

no extremum @ \( x = 0 \) (it's a discontinuity).

\[
\text{conc. up on } (-\infty, 0) \& (0, 6)
\]

\[
\text{conc. down on } (6, \infty)
\]

Inflection pt. @ \( x = 6 \), \( y = f(6) = -\frac{4}{36} = -1/9 \).

(sketch on next page)
Sketch: (not exactly to scale)

incl/dec.

V.A. \( x = 0 \)

H.A. \( y = 0 \)

(4, -1/8) local min

(6, 1/8) inflection pt.

concavity:
7. [12 points] A 15 foot ladder is leaning against a wall, and sliding down the side. At this moment, the bottom of the ladder is 9 feet away from the wall, and is sliding away at 2 feet per second. Determine how quickly the top of the ladder is sliding down the wall at this moment (in feet per second).

At key moment

\[ x^2 + y^2 = 15^2 \]

\[ \frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(15^2) \]

\[ 2x \cdot x' + 2y \cdot y' = 0 \]

\[ \Rightarrow \text{At key moment,} \]

\[ 2 \cdot 9 \cdot 2 + 2 \cdot 12 \cdot y' = 0 \]

\[ 24y' = -36 \]

\[ y' = -\frac{36}{24} = -\frac{3}{2} \]

(Req. since decreasing as ladder falls).

The top of the ladder is falling at \( \frac{3}{2} \) foot per second.
8. [12 points] A small rectangular box with a square base with no lid is to be constructed out of 12 square inches of cardboard. Determine what dimensions the box should have in order for its volume to be as large as possible.

Cardboard needed:
\[ x^2 + 4xy = 12 \]

We want to maximize volume:
\[ V = x^2y. \]

Solve for \( y \) using the constraint:

\[ 4xy = 12 - x^2 \]
\[ y = \frac{12 - x^2}{4x} \]

So volume, as a function of \( x \), is
\[ V(x) = x^2 \left( \frac{12 - x^2}{4x} \right) = \frac{1}{4}x(12 - x^2) \]

Feasible values: we need \( x > 0 \)
\& \( y > 0 \Rightarrow \frac{12 - x^2}{4x} > 0 \Rightarrow 12x^2 > x \Rightarrow x < \sqrt{12} = 2\sqrt{3} \)

So the interval is \([0, 2\sqrt{3}]\).

We can maximize with the closed interval method:
\[ V(x) = \frac{1}{4}x(12 - x^2) \]
\[ V'(x) = \frac{1}{4}(12 - 3x^2) = \frac{3}{4}(4 - x^2) \]

\[ \Rightarrow \text{opt. pts @ } x = \pm 2; \text{ discard } -2 \text{ since it's out of the interval.} \]

Check endpoints & \( x = 2 \):
\[ V(0) = \frac{1}{4} \cdot 0 \cdot 12 = 0 \]
\[ V(2) = \frac{1}{4} \cdot 2 \cdot 8 = 4 \leftarrow \text{ max} \]
\[ V(2\sqrt{3}) = \frac{1}{4} \cdot 2\sqrt{3} \cdot 0 = 0 \]

So the max volume is 4, which occur for \( x = 2, \ y = \frac{12 - 2^2}{4 \cdot 2} = 1 \).
9. Consider the following piecewise function.

\[ f(x) = \begin{cases} 
2 & x < 0 \\
\frac{1}{x-2} & 0 \leq x < 2 \\
2\sqrt{x-2} & x \geq 2 
\end{cases} \]

(a) [6 points] Sketch the graph \( y = f(x) \) on the axes below (for this sketch, you don’t need to take any derivatives or apply the techniques from Chapter 3; instead think about how each piece of the graph is obtained from a graph you already know about).

(b) [6 points] Evaluate each of the following quantities (no explanation or scratchwork is required).

- \( \lim_{x \to 0^-} f(x) = 2 \)
- \( \lim_{x \to 2^-} f(x) = -\infty \)
- \( \lim_{x \to 0^+} f(x) = -\frac{1}{2} \)
- \( \lim_{x \to 2^+} f(x) = 0 \)
- \( f(0) = -\frac{1}{2} \)
- \( f(2) = 0 \)
The definition of $f(x)$ is reproduced below for convenience.

$$f(x) = \begin{cases} 
2 & x < 0 \\
\frac{1}{x-2} & 0 \leq x < 2 \\
2\sqrt{x - 2} & x \geq 2
\end{cases}$$

(c) [3 points] Determine all points where $f(x)$ is discontinuous.

$x = 0$ is a jump discontinuity

$$\left( \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \right)$$

$x = 2$ is an infinite discontinuity

$$\left( \lim_{x \to 2^-} f(x) = -\infty \right)$$

There are no other discontinuities; the three functions used are continuous on the intervals where they apply.

(continued on reverse)
10. [12 points] Consider the function \( f(x) = \frac{x}{4 + x^2} \).

(a) Compute \( f'(x) \), and simplify.

\[
f'(x) = \frac{1 \cdot (4 + x^2) - x \cdot 2x}{(4 + x^2)^2}
\]

\[
= \frac{4 - x^2}{(4 + x^2)^2}
\]

\[
= \frac{(x + 2)(x - 2)}{(4 + x^2)^2}
\]

(b) Determine all critical numbers of \( f(x) \).

\[ f'(x) = 0 \text{ if } (x + 2)(x - 2) = 0 \]

\[ x = \pm 2 \]

Denominator is never 0 since \( 4 + x^2 > 0 \).

Num. is 0 at \( x = \pm 2 \).
(c) Determine the intervals on which \( f(x) \) is increasing and decreasing.

\[
\begin{array}{c|c|c|c}
\text{x} & -2 & 2 \\
\hline
\frac{1}{(4+x^3)} & - & - & - \\
\hline
x^2 - & - & + & + \\
\hline
\frac{1}{x} & - & + & - \\
\hline
f & \searrow & \nearrow & \searrow \\
\end{array}
\]

dec. on \((-\infty, -2) \& (2, \infty)\)
inc. on \((-2, 2)\)

(d) Classify each critical point as a local minimum, a local maximum, or neither.

\( x = -2 \) is a local min.
by 1st deriv. test \((\downarrow\uparrow)\)

\( x = 2 \) is a local max.
by 1st deriv. test \((\nearrow\searrow)\).