Amherst College
Department of Mathematics and Statistics

Math 105 | Midterm 3 | Fall 2018

Name: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	Total
Points:	9	9	10	12	20	60
Score:						

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1. [9 points] Find all of the **critical numbers** of each function.

(a) $(x-1)(x-5)^3$

$$f'(x) = 1\cdot(x-5)^3 + (x-1)\cdot 3(x-5)^2$$
$$= (x-5)^2\cdot\left[(x-5)+3(x-1)\right]$$
$$= (x-5)^2\cdot[4x-8]$$
$$= 4(x-5)^2(x-2)$$

never undefined; $f'(x)=0$ when $\boxed{x=2 \text{ or } x=5}$.

(b) $10x^{4/5} - 5x^{9/5}$

$$f'(x) = 10\cdot\frac{4}{5}x^{-1/5} - 5\cdot\frac{9}{5}x^{4/5}$$
$$= 8x^{-1/5} - 9x^{4/5} = \frac{8}{x^{1/5}} - 9\cdot x^{4/5}\cdot\frac{x^{1/5}}{x^{1/5}} = \frac{8-9x}{x^{1/5}}$$

undefined @ $\underline{x=0}$ (divis. by 0).

equal to 0 when $8-9x=0$, i.e. $\underline{x=8/9}$.

$$\boxed{x=0 \quad \& \quad x=8/9}$$

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2. [9 points] Evaluate each **limit at infinity**.

(a) $\lim\limits_{x\to\infty} \dfrac{3x^2-8x+4}{6x^2+7x-4} \cdot \dfrac{1/x^2}{1/x^2}$

$$= \lim_{x\to\infty} \frac{3-8/x+4/x^2}{6+7/x-4/x^2} = \frac{3-8/\infty+4/\infty}{6+7/\infty-4/\infty} = \frac{3-0+0}{6+0-0}$$

$$= \frac{3}{6} = \boxed{\frac{1}{2}}$$

(b) $\lim\limits_{x\to-\infty} \dfrac{7x+1}{2x^2-3x-7} \cdot \dfrac{1/x^2}{1/x^2}$

$$= \lim_{x\to-\infty} \frac{7/x+1/x^2}{2-3/x-7/x^2} = \frac{7/(-\infty)+1/\infty}{2-3/(-\infty)-7/\infty}$$

$$= \frac{0+0}{2-0-0} = \frac{0}{2} = \boxed{0}$$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2+7}{2x+3} \cdot \dfrac{1/x}{1/x} = \lim\limits_{x\to\infty} \dfrac{x+7/x}{2+3/x} = \dfrac{\infty+0}{2+0} = \dfrac{\infty}{2} = \boxed{\infty}$

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3. [10 points] Find the **absolute maximum and minimum** values of $f(x) = x\sqrt{12-x}$ on the interval $[3, 11]$.

$$f'(x) = 1 \cdot \sqrt{12-x} + x\frac{1}{2\sqrt{12-x}} \cdot (-1)$$

$$= \sqrt{12-x} - \frac{x}{2\sqrt{12-x}}$$

$$= \frac{\sqrt{12-x}}{1} \cdot \frac{2\sqrt{12-x}}{2\sqrt{12-x}} - \frac{x}{2\sqrt{12-x}}$$

$$= \frac{2(12-x) - x}{\sqrt{12-x}} = \frac{24-3x}{\sqrt{12-x}}$$

crit. numbers: when num. or denom is 0,

i.e. when $24-3x = 0 \iff x = 8$ ← crit num.

or $\sqrt{12-x} = 0 \iff x = 0$ ← not in $[3,11]$, so ignore.

candidates in $[3,11]$: 3, 8, & 11 (crit num. & boundaries).

$f(3) = 3\sqrt{9} = 9$ ← abs. min

$f(8) = 8\sqrt{4} = 16$ ← abs. max

$f(11) = 11\sqrt{1} = 11$

min value 9 @ $x=3$
max value 16 @ $x=8$

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4. [12 points] Sand is being poured onto a conical pile at a rate of 40 cubic feet per minute. The diameter of the pile is 4 times the height. How quickly (in feet per minute) is the radius of the pile increasing, when the radius is equal to 20 feet?

(The volume of a cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.)

diameter = 4 x height means $2r = 4h$, ie. $h = \frac{1}{2}r$.

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi r^2 \cdot \frac{1}{2} r$$
ie. $V = \frac{1}{6}\pi r^3$

@ specific moment, we know:

$V' = 40$

$r = 20$

& we want r'.

differentiating:

$$V' = \frac{1}{6}\pi \cdot 3r^2 \cdot r'$$
$$V' = \frac{1}{2}\pi r^2 \cdot r'$$

so at the specific moment:

$$40 = \frac{1}{2}\pi \cdot 20^2 \cdot r'$$
$$40 = 200\pi \cdot r'$$
$$\Rightarrow r' = \frac{40}{200\pi} = \frac{1}{5\pi}.$$

The radius is growing by $\boxed{\frac{1}{5\pi} \text{ (feet per minute)}}$.

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5. [20 points] Consider the function

$$f(x) = \frac{x^2+4}{x^2+12}.$$

The first two derivatives of this function are as follows. **You do not need to compute these.**

$$f'(x) = \frac{16x}{(x^2+12)^2}$$

$$f''(x) = -\frac{48(x^2-4)}{(12+x^2)^3}$$

(a) On which intervals is $f(x)$ **increasing** on which intervals is it **decreasing**?

num. is 0 @ x=0 } of f'(x)
denom. never 0.

	$x<0$	$x>0$
$16x$	−	+
$\frac{1}{(x^2+12)^2}$	+	+
f'	−	+
f	↘	↗

dec. on $(-\infty,0)$
inc. on $(0,\infty)$

(b) Using your answer to part (a), determine any **local max(s) and/or local min(s)** of $f(x)$. Give **both the** x **and** y **coordinates.**

1st deriv. test: local min. @ $x=0$,
$y = f(0) = \frac{0+4}{0+12} = 1/3$

local min @ $(0, \frac{1}{3})$

(continued on reverse)

For convenience, the function and its derivatives are written again below.

$$f(x) = \frac{x^2+4}{x^2+12}$$
$$f'(x) = \frac{16x}{(x^2+12)^2}$$
$$f''(x) = -\frac{48(x^2-4)}{(12+x^2)^3}$$

(c) On which intervals is $f(x)$ **concave up** and on which intervals is it **concave down**?

$$f''(x) = -\frac{48(x+2)(x-2)}{(12+x^2)^3}$$

num. of $f''(x)$ is 0 @ $x=\pm 2$ (where $x^2=4$)
denom. is never 0.

	$x<-2$	$-2<x<2$	$x>2$
$x+2$	−	+	+
$x-2$	−	−	+
$-\frac{48}{(12+x^2)^3}$	−	−	−
f''	−	+	−
f	∩	∪	∩

conc. down on $(-\infty,-2)$ & $(2,\infty)$
conc. up on $(-2,2)$.

(d) Using your answer to part (c), determine any **point(s) of inflection** of $f(x)$. Give **both the x and the y coordinates.**

$x=\pm 2$ (where infl. changed), $y = f(\pm 2) = \frac{(\pm 2)^2+4}{(\pm 2)^2+12}$

$= \frac{4+4}{4+12} = \frac{8}{16} = \frac{1}{2}.$

inflection pt. @ $(-2,\frac{1}{2})$ & $(2,\frac{1}{2})$.

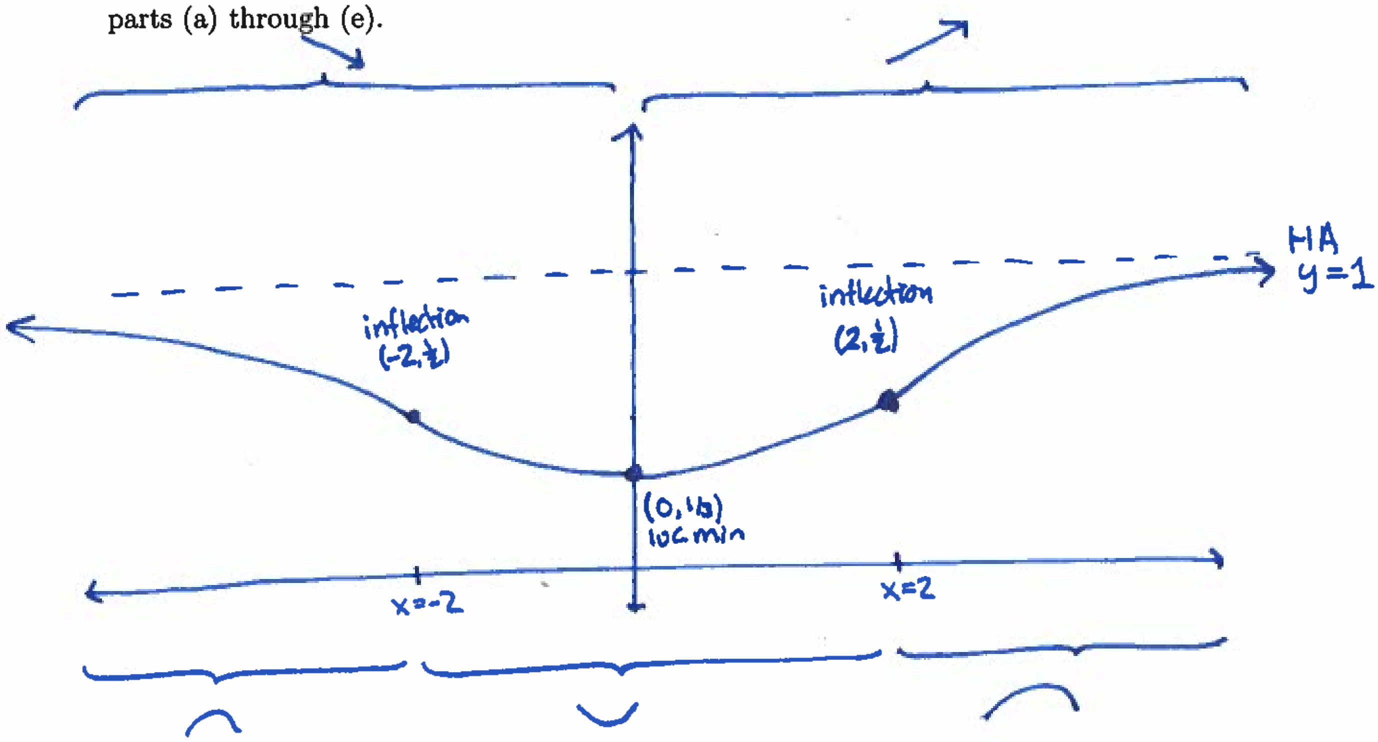
(e) Determine any **horizontal asymptotes** of the function $f(x) = \frac{x^2+4}{x^2+12}$.

$$\lim_{x\to\infty} \frac{x^2+4}{x^2+12} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1+4/x^2}{1+12/x^2} = \frac{1+4/\infty}{1+12/\infty} = \frac{1+0}{1+0} = 1$$

& similarly $\lim_{x\to-\infty} \frac{x^2+4}{x^2+12} = \lim_{x\to\infty} \frac{1+4/x^2}{1+12/x^2} = \frac{1+4/\infty}{1+12/\infty} = \frac{1+0}{1+0} = 1$

horiz. asymptote @ $y=1$ (in both directions)

(f) Draw a rough sketch of the graph $y = f(x)$, incorporating the information you found in parts (a) through (e).



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